Valuation of Commodity assets and the option to invest in the presence of stochastic prices and incomplete information

Mondher Bellalah [1]

This article extends the three models in Schwartz (1997) to describe the stochastic behavior of commodity prices in the presence of mean reversion and shadow costs of incomplete information. The implications of the models are studied with respect to the valuation of financial and real assets. We extend the analysis in Schwartz (1997) to account for the e.ects of shadow costs of incomplete information as defined in Merton (1987).
The first one-factor model assumes that the logarithm of the spot commodity price follows a mean reverting process. The second model is a two-factor model in which the convenience yield is stochastic. The third model accounts for stochastic interest rates. The implications of the models are studied for capital budgeting decisions.
We develop also a one-factor model for the stochastic behavior of commodity prices which preserves the main properties of more complex two-factor models. When applied for the valuation of long-term commodity projects, the model gives practically the same results as more complex models.

[1]Professor of finance, Universities of Cergy and Paris-Dauphine.
Correspondence to : Mondher Bellalah, University of Cergy, THEMA, 33 boulevard du Port, 95 011 Cergy, France, e-mail : bellalah@u-cergy.fr

We thank professors Richard Roll, Hayne Leland and Dean Paxson for their helpful discussions on information costs.

1 Introduction

The stochastic behavior of commodity prices plays a crucial role in the pricing of commodity derivatives and in capital budgeting decisions. Earlier studies are based on constant interest rates and convenience yields in the pricing of financial and real commodity derivatives. This assumption implies that the distribution of future spot prices has a variance that increases without bound as the horizon increases.

This article uses and compares three models of the stochastic behavior of commodity prices in the presence of shadow costs of incomplete information. In the first model, the logarithm of the spot commodity price is assumed to follow a mean reverting process of the Ornstein-Uhlenbeck type. In the second model, the convenience yield is also assumed to follow a mean reverting process. In the third model, the interest rate is assumed to follow mean reverting process. Closed-form solutions are derived in these three models for forward and futures contracts.

The implications of the model are studied for the term structure of futures prices and for hedging contracts for future delivery.

The real options methodology to investment under uncertainty and in particular, the determination of optimal investment rules depend on the stochastic process for the underlying commodity. The value and the investment rules are determined in the context of the three models by accounting for shadow

costs of incomplete information.
These costs are defined as in Merton (1987). For an introduction to the basic concepts for the pricing of derivative assets and real options under uncertainty
and incomplete information, we can refer to Bellalah and Jacquillat (1995), Bellalah (1999, 20001). The application of option concepts to value real assets such as copper mines and oil deposits has been successful because of the existence of well-developed futures markets for these commodities. These markets allow the extraction of the essential information.
The traditional approach for the valuation of investment projects is the net present value approach. An alternative approach is the certainty-equivalent approach which avoids the computation of a risk-adjusted discount factor, using instead the relevant risk-free rate of interest.
2
Brennan and Schwartz (1985) apply the option pricing theory to value investment projects in natural resources where the spot price of the commodity follows a geometric Brownian motion. The option pricing theory uses the information contained in futures prices since these prices are used in the estimation
of the convenience yield. The approach is based on the use of the risk free rate rather than a risk-adjusted discount rate and allows for managerial flexibility in the form of options.
Schwartz (1997) compared three models of the stochastic behavior of commodity prices : a one-factor model, a two-factor model and a three-factor model.
Schwartz (1998) develops a one-factor model that preserves the main characteristics
of two-factor models. We extend the analysis in these two papers to account for the e.ects of incomplete information as it appears in the models of Merton (1987) and Bellalah (2001).
The paper is organized as follows.
Section 1 presents the valuation models.
Section 2 looks at the implications of the di.erent models for investment under uncertainty.
Section 3 presents the valuation models and the long term model. An application is provided for the valuation of European options.
Section 4 compares the simple model and the two-factor model with respect to their optimal exercise criteria.

1. The Valuation Models for commodity futures under incomplete information

This section presents three models of commodity prices and the formulas for futures contracts. The models allow for closed form solutions for futures prices.

A. Model 1

Schwartz (1997) assumed that the commodity spot price follows the stochastic process

$dS = ê(μ - \ln S)Sdt + óSdz$ (1)

3

where dz is an increment to a standard Brownian motion and ê refers to the speed of adjustment.

When $X = \ln S$, applying Ito’s Lemma allows to characterize the log price by an Ornstein-Uhlenbeck stochastic process

$dX = ê(á - X)dt + ódz$ (2)

with

$$á = μ - \frac{ó^2}{2ê} \quad (3)$$

where ê measures the degree of mean reversion to the long run mean log price á.

Under standard assumptions, Schwartz (1997) gives the following dynamics of the Ornstein-Uhlenbeck stochastic process under the equivalent martingale measure

$dX = ê(á_. - X)dt + ódz_.$ (4)

where $á_. = á - ë$ where ë is the market price of risk.

From equation (4), the conditional distribution of X at time T under the equivalent martingale measure is normal. The mean of X is

$E_0[X(T)] = e^{-êT}X(0) + (1 - e^{-êT})á_.$

The variance of X is

$$V ar_0[X(T)] = \frac{ó^2}{2ê}(1 - e^{-2êT}) \quad (5)$$

When the interest rate is constant, the futures or the forward price of the commodity corresponds to the expected price of the commodity for the maturity T.

Using the properties of the log-normal distribution, the futures or the forward price is given by

$$F(S, T) = E[S(T)] = \exp(E_0[X(T)] + \frac{1}{2} V ar_0[X(T)]) \quad (6)$$

and

$$F(S, T) = \exp(e^{-êT} \ln S + (1 - e^{-êT})á_. + \frac{ó^2}{4ê}(1 - e^{-2êT})) \quad (7)$$

4

This equation can be written in a log form as

$$\ln F(S, T) = e^{-êT} \ln S + (1 - e^{-êT})á_. + \frac{ó^2}{4ê}(1 - e^{-2êT}) \quad (8)$$

Equation (7) is solution to the partial di.erential equation

1 2

$ó_2 S_2 F_{SS}$ + ê(µ - ë - ln S)SF_S - F_T = 0 (9)

under the terminal boundary condition F(S, 0) = S.

B. Model 2

In this two-factor model, the first factor corresponds to the spot price of the commodity with the following dynamics

dS = (µ - ä)Sdt + $ó_1 Sdz_1$ 10)

where ä is the instantaneous convenience yield which can be seen as the flow of services accruing to the holder of the commodity rather than the buyer of the futures contract.

The second factor corresponds to the convenience yield with the following dynamics

dä = ê(á - ä)dt + $ó_2 dz_2$ (11)

where

$dz_1 dz_2$ = ñdt (12)

. Hence, equation (10) allows for a stochastic convenience yield, which follows an Ornstein-Uhlenbeck stochastic process.

When ä is a deterministic function of S, ä(S) = ê ln S, this model reduces to model 1.

When ä is a constant, this model reduces to the Brennan and Schwartz (1985).

When X = lnS, applying Ito's Lemma allows to characterize the log price as

dX = (µ - ä -

1 2

$ó_2$

$_1$)dt + $ó_1 dz_1$ (13)

The commodity is viewed as an asset paying a stochastic dividend yield ä and the risk adjusted drift of the commodity is (r+$ë_S$ -ä) where $ë_S$ refers to

5

an information cost for the asset S. In fact, we can show as in Bellalah (2001) that under the equivalent martingale measure, the stochastic processes for the two factors can be written as

dS = (r + $ë_S$ - ä)Sdt + $ó_1 Sdz_{.1}$ (14)

dä = [ê(á - ä) - ë]dt + $ó_2 dz_{.2}$ (15)

$dz_{.1} dz_{.1}$ = ñdt (16)

where ë refers in this model to the market price of convenience yield risk.

Using the same approach as in Bellalah (2001), Futures prices satisfy the following PDE

1 2

$ó_2$

$_1 S_2 F_{SS}$+

1

2

$ó_2$

$_2 F_{ää}$+$ó_1 ó_2$ñ$SF_{Sä}$+(r+$ë_S$-ä)SF_S+[ê(á-ä)-ë]$F_ä$-F_T = 0 (17)

under the terminal boundary condition F(S, ä, 0) = S.
As in Schwartz (1997), the solution given is given by

$$F(S, ä, T) = S \exp\left[-ä \frac{1 - e^{-êT}}{ê} + A(T)\right] \quad (18)$$

This can be written in a log form as

$$\ln F(S, ä, T) = \ln S - ä \frac{1 - e^{-êT}}{ê} + A(T) \quad (19)$$

where

$$A(T) = \left(r + ë_S - \hat{á} + \frac{1}{2}\frac{ó_2^2}{ê^2} - \frac{ó_1 ó_2 ñ}{ê}\right)T + \frac{1}{4} ó_2^2 \frac{1 - e^{-2êT}}{ê^3} + \left(\hat{á}ê + ó_1 ó_2 ñ - \frac{ó_2^2}{ê}\right)\frac{1 - e^{-êT}}{ê^2}$$

and

$$\hat{á} = á - \frac{ë}{ê} \quad (20)$$

The main di.erence between this solution and that in Schwartz concerns the discount rate in A(T) which appears to be the interest rate plus the information cost on the asset S rather than the interest rate only.

C. Model 3

In this three-factor model, the three factors are the spot price of the commodity, the instantaneous convenience yield, and the instantaneous interest rate. When the interest rate follows a mean reverting process as in Vasicek (1977), using equations (14) and (15), the joint stochastic process for the three factors under the equivalent martingale measure can be written as

$$dS = (r + ë_S - ä)S\,dt + ó_1 S\,dz_{.1} \quad (21)$$

$$dä = ê(\hat{á} - ä)\,dt + ó_2\,dz_{.2} \quad 22)$$

$$dr = a(m^* - r)dt + \sigma_3 dz_3^* \quad (23)$$

where
$$dz_1^* dz_2^* = \rho_1 dt, \quad dz_2^* dz_3^* = \rho_2 dt, \quad dz_1^* dz_3^* = \rho_3 dt \quad (24)$$

where a and m^* refer respectively to the speed of adjustment coefficient and the risk adjusted mean short rate of the interest rate process.

In this context, futures prices must satisfy the following PDE

$$\frac{1}{2}\sigma_1^2 S^2 F_{SS} + \frac{1}{2}\sigma_2^2 F_{\delta\delta} + \frac{1}{2}\sigma_3^2 F_{rr} + \sigma_1\sigma_2\rho_1 S F_{S\delta} + \sigma_2\sigma_3\rho_2 F_{\delta r} + \sigma_1\sigma_3\rho_3 S F_{Sr} + (r+\lambda_S - \delta) S F_S + [\kappa(\hat{\alpha} - \delta)]F_\delta + a(m^* - r)F_r - F_T = 0 \quad (25)$$

under the terminal boundary condition $F(S, \delta, r, 0) = S$.

Following the analysis in Schwartz (1997), the solution is given by :

$$F(S, \delta, r, T) = S \exp\left[-\delta\frac{(1 - e^{-\kappa T})}{\kappa} + \frac{(r + \lambda_S)(1 - e^{-aT})}{a} + C(T)\right] \quad (26)$$

This can be written in a log form as

$$\ln F(S, \delta, r, T) = \ln S - \frac{\delta(1 - e^{-\kappa T})}{\kappa} + \frac{(r + \lambda_S)(1 - e^{-aT})}{a} + C(T)] \quad (27)$$

where

$$C(T) = \frac{(\kappa\hat{\alpha} + \sigma_1\sigma_2\rho_1)[(1 - e^{-\kappa T}) - \kappa T]}{\kappa^2} - \frac{\sigma_2^2(4(1 - e^{-\kappa T}) - (1 - e^{-2\kappa T}) - 2\kappa T)}{4\kappa^3} - \frac{(am^* + \sigma_1\sigma_3\rho_3)[(1 - e^{-aT}) - aT]}{a^2} - \frac{\sigma_3^2(4(1 - e^{-aT}) - (1 - e^{-2aT}) - 2aT)}{4a^3}$$

$$+ó_2ó_3ñ_2\frac{(1 - e^{-êT}) + (1 - e^{-aT}) - (1 - e^{-(ê+a)T})}{êa(ê + a)} + \frac{ê^2(1 - e^{-aT}) + a^2(1 - e^{-êT}) - êa^2T - aê^2T}{ê^2a^2(ê + a)} \quad (28)$$

As it is well known, in the presence of stochastic interest rates, forward prices are di.erent from futures prices. The present value of a unit discount bond payable at time T is given in Vasicek (1977) as

$$B(r, T) = \exp\left[-r\frac{(1 - e^{-aT})}{a} + \frac{m.((1 - e^{-aT}) - aT)}{a} - \frac{ó_3^2(4(1 - e^{-aT}) - (1 - e^{-2aT}) - 2aT)}{4a^3}\right] \quad (29)$$

The present value of a forward commitment to deliver one unit of the commodity, P(S, ä, r, T) is solution to the PDE under boundary conditions identical to equation (25) except that in the right-hand side, rP replaces zero. The solution is

$$P(S, ä, r, T) = S\exp\left[-ä\frac{(1 - e^{-êT})}{ê} + D(T)\right] \quad (30)$$

where

$$D(T) = \frac{(ê\hat{á} + ó_1ó_2ñ_1)[(1 - e^{-êT}) - êT]}{ê^2} - \frac{ó_2^2(4(1 - e^{-êT}) - (1 - e^{-2êT}) - 2êT)}{4ê^3} \quad (31)$$

Equation (30) gives the present value of a forward commitment. Equation (31) gives the present value of a unit discount bond. The forward price implied by model 3 is obtained by dividing P(S, ä, r, T) by B(r, T).

2. Investment Under Uncertainty and the value of the option to invest

The dynamics of commodity prices present several implications for project valuation (like mines, oil deposits, etc.) and the search of the optimal investment rule. This rule refers to the commodity price above which it is optimal

to undertake the project immediately.

Example

Consider a copper mine that can produce one ounce of copper at the end of

each year for 10 years. The initial investment K = 2 and the unit cost of
production C = 0.40. Assume that once the investment is done, production
will go ahead for the next 10 years.
The first step determines the net present value of the project and the second
step values the option to invest.
The NPV once the investment has been decided is
NPV = Ó$_{10}$
$_{T=1}$P(r, T, .) - CÓ$_{10}$
$_{T=1}$B(r, T) - K (32)
where
P(r, T, .) : present value of the commodity to be received at time T when
the interest rate is r,
B(r, T) : present value of one dollar to be received at time T when the interest
rate is constant, e$_{-rT}$.
Discounted Cash Flow Criteria
The DCF approach needs the specification of the discount rate and the expected
spot copper prices for the next ten years. In practice, spot prices are
assumed constant. The project's value is very sensitive to the discount rate
used.
Constant Convenience Yield : Model 0
In the standard real option approach, instead of discounting at a risk-adjusted
rate, certainty equivalent cash flows are discounted at the riskless rate.
In a constant convenience yield model, Model 0, the spot commodity follows
the process
dS
S
= (r + ë$_{S}$ - c)dt + ódz$_{.}$ (33)
where c is the constant convenience yield to distinguish it from ä used in the
stochastic convenience yield models.
The NPV (32) becomes
NPV (S) = SÓ$_{10}$
$_{T=1}$e$_{-cT}$ - CÓ$_{10}$
$_{T=1}$e$_{-rT}$ - K = Sâ$_{1}$ - â$_{2}$ (34)
9
As in Bellalah (2001), we can show that the option to invest V (S) satisfies
the ordinary di.erential equation
1 2
ó$_{2}$S$_{2}$V$_{SS}$ + (r + ë$_{S}$ - c)SV$_{S}$ - (r + ë$_{V}$)V = 0 (35)
under the boundary condition
V (S) ¡Ý max[Sâ$_{1}$ - â$_{2}$, 0] (53)
where ë$_{V}$ refers to an information cost on asset V . The solution to this
equation is
V (S) = (S.â$_{1}$ - â$_{2}$)(
S
S.
)$_{d}$

where the commodity price above which it is optimal to invest in the project is given by

$$S^{*} = \left(\frac{\beta_2 d}{\beta_1 (d - 1)} \right)$$

$$d = \frac{1}{2} - \frac{(r - c + \lambda_V)}{\sigma^2} + \sqrt{\left[\frac{1}{2} - \frac{(r - c + \lambda_V)}{\sigma^2}\right]^2 + \frac{2(r + \lambda_S)}{\sigma^2}} \quad (37)$$

Mean Reverting Spot Price : Model 1

The NPV in Model 1 is computed using equation (32) by discounting the prices given by equation (7).

$$NPV = \sum_{T=1}^{10} P(r, T, .) - C \sum_{T=1}^{10} B(r, T) - K \quad (32)$$

where

$P(r, T, .)$: present value of the commodity to be received at time T when the interest rate is r,

$B(r, T)$: present value of one dollar to be received at time T when the interest rate is constant, e^{-rT}.

with

$$P(r, T, .) = e^{-(r+\lambda_S)T} F(S, T)$$

and

$$F(S, T) = \exp\left(e^{-\kappa T} \ln S + (1 - e^{-\kappa T})\alpha^{*} + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa T})\right)$$

The value of the investment option, $V(S)$ can be obtained by solving a PDE identical to equation (9) in which in the right-hand side we have $(r + \lambda_V)V$ instead of zero.

The boundary condition is the maximum of the NPV in this case and zero. This can be written as

$$\frac{1}{2}\sigma^2 S^2 V_{SS} + \kappa(\mu - \lambda - \ln S) S V_S - V_T = (r + \lambda_V)V$$

under the terminal boundary condition

$$V(S, 0) = Max[NPV, 0]$$

Stochastic Convenience Yield : Model 2

The NPV in Model 2 is computed using equation (32) which depends on the spot price, the convenience yield and the present value of one unit of commodity (which is obtained by discounting the future or forward price in equation (18).

NPV = Ó$_{10}$
$_{T=1}$P(r, T, .) - CÓ$_{10}$
$_{T=1}$B(r, T) - K (32)

where

P(r, T, .) : present value of the commodity to be received at time T when the interest rate is r,

B(r, T) : present value of one dollar to be received at time T when the interest rate is constant, e$_{-rT}$.

with

P(r, T, .) = e$_{-(r+ës)T}$F(S, ä, T)

F(S, ä, T) = S exp[-ä
1 - e$_{-êT}$
ê
+ A(T)] (18)

The value of the option to invest V (S, ä) satisfies a PDE identical to equation (17), except that the right-hand side is (r + ë$_V$)V instead of zero.

1 2
ó$_2$
$_1$S$_2$V$_{SS}$+
1
2
ó$_2$
$_2$S$_2$V$_{ää}$+ó$_1$ó$_2$ñSV$_{Sä}$+(r+ë$_S$-ä)SV$_S$+[ê(á-ä)-ë]V$_ä$-V$_T$ = (r+ë$_V$)V

under the terminal boundary condition

V (S, ä, 0) = Max[NPV, 0]

11

Stochastic Convenience Yield and Interest rates : Model 3

The NPV in Model 3 is computed using equation (32). It depends on the spot price of the commodity, the convenience yield and the interest rate.

NPV = Ó$_{10}$
$_{T=1}$P(r, T, .) - CÓ$_{10}$
$_{T=1}$B(r, T) - K (32)

where

P(r, T, .) : present value of the commodity to be received at time T when the interest rate is r,

B(r, T) : present value of one dollar to be received at time T when the interest rate is constant, e$_{-rT}$.

The present value of a unit of the commodity is computed using equation (30)

P(S, ä, r, T) = S exp[-ä(1 - e$_{-êT}$)
ê
+ D(T)] (30)

where

$$D(T) = \frac{[ê\hat{á} + ó_1ó_2ñ_1)(1 - e^{-êT}) - êT]}{ê^2} - \frac{ó_2^2(4(1 - e^{-êT}) - (1 - e^{-2êT}) - 2êT)}{4ê^3} \quad (31)$$

The present value of a unit discount bond is computed using equation (29).

$$B(r, T) = \exp[-r\frac{(1 - e^{-aT})}{a} + \frac{m.((1 - e^{-aT}) - aT)}{a} - \frac{ó_3^2(4(1 - e^{-aT}) - (1 - e^{-2aT}) - 2aT)}{4a^3}] \quad (29)$$

The value of the option to invest V (S, ä, r) satisfies a PDE identical to (25) except that the right-hand side is $(r + ë_V)V$ instead of zero.

$$\frac{1}{2}ó_1^2S^2V_{SS} + \frac{1}{2}ó_2^2V_{ää} + \frac{1}{2}ó_3^2V_{rr} + ó_1ó_2ñ_1SV_{Sä} + ó_2ó_3ñ_2V_{är} + ó_1ó_3ñ_3SV_{Sr} + (r+ë_S - ä)SV_S + [ê(\hat{á} - ä)]V_ä + a(m. - r)V_r - V_T = (r+ë_V)V$$

under the terminal boundary condition

$$V(S, ä, r, 0) = Max[NPV, 0]$$

3. The Valuation Models, the long term model and European option pricing

This section presents first the basic constant-convenience-yield model and the two-factor stochastic convenience-yield model. Then, the basic model is adjusted to account for the important features of the two-factor model. This model is referred to as the ”long-term model”.

A. The basic Model

Schwartz (1998) assumed that the commodity spot price under the equivalent martingale measure is given by

$$\frac{dS}{S} = (r + ë_S - c)Sdt + ódz \quad (38)$$

where dz is an increment to a standard Brownian motion, r is the interest rate, ó is the volatility of the rate of return and c is a constant convenience yield.

The futures price F with a maturity T for a spot asset S is given by

$F(S, T) = Se^{(r+ë_S-c)T}$ (39)

Applying Ito's Lemma to equation (39) shows that the volatility of the futures returns $\frac{dS}{S}$ is equal to ó.

The value of a contingent claim V (S, T) must satisfy the following partial di.erential equation

$\frac{1}{2}ó^2S^2V_{SS} + (r + ë_S - c)SV_S - V_T - (r + ë_V)V = 0$ (40)

under the appropriate boundary conditions.

If the contingent claim represents a project, then the cash flows on the project CF(T) must be added to equation (40).

B. The Two-Factor Model

In this two-factor model, the first factor corresponds to the spot price of the commodity with the following dynamics

$dS = (r + ë_S - ä)Sdt + ó_1Sdz_1$ (41)

where ä is the instantaneous stochastic convenience yield. It can be seen as the flow of services accruing to the holder of the commodity rather than the buyer of the futures contract.

The second factor corresponds to the stochastic convenience yield with the following dynamics

$dä = ê(\hat{á} - ä)Sdt + ó_2dz_2$ (42)

where

$dz_1dz_2 = ñdt$ (43)

In this formulation, the magnitude of the speed of adjustment ê > 0 measures the degree of mean reversion to the long-run mean convenience yield á.

Futures prices F(S, ä, T) are given by

$F(S, ä, T) = S \exp[-ä\frac{1 - e^{-êT}}{ê} + A(T)]$ (44)

where

$A(T) = (r+ë_S-\hat{á}+\frac{1}{2}\frac{ó_2^2}{ê^2} - \frac{ó_1ó_2ñ}{ê})T+\frac{1}{4}ó_2$

2
1 - e$_{-2êT}$
ê$_3$ +(ˆáê+ó$_1$ó$_2$ñ-
ó$_2$
$_2$ ê
)
1 - e$_{-2êT}$
ê$_3$
and where the risk-adjusted long-run mean of the convenience yield process
is given by
ˆá = á -
ë ê
(45)
where ë stands for the market price of convenience yield risk.
Applying Ito’s Lemma to equation (44), we can show that the variance
of the futures returns depends only on the time to maturity of the futures
contract
ó$_2$
$_F$T = ó$_2$
$_1$ + ó$_2$
2
(1 - e$_{-êT}$)
ê - 2ñó$_1$ó$_2$
(1 - e$_{-êT}$)
ê
(46)
When the maturity of the futures contract tends to infinity, this variance
converges to a fixed value
ó$_2$
$_F$ (□‡) = ó$_2$
$_1$ +
ó$_2$
2
ê -
2ñó$_1$ó$_2$
ê
(47)
14
The value of any contingent claim must satisfy the following PDE
1 2
ó$_2$
$_1$S$_2$V$_{SS}$+
1
2
ó$_2$
$_2$V$_{ää}$+ó$_1$ó$_2$ñSV$_{Sä}$+(r+ë$_S$-ä)SV$_S$+[ê(ˆá-ä)]V$_ä$-V$_T$ -(r+ë$_V$)V = 0 (48)
under the appropriate terminal boundary conditions.

C. The Long-Term Model

Given equations (41) and (43), the risk-neutral distribution of spot prices is log-normal with mean equal to the forward price in equation (44). The variance can be obtained by integrating the variance in equation (46).

The objective is to develop a model which matches the term structure of futures prices and volatilities implied by the two-factor model.

When maturity increases in the two-factor model, the rate of change in the futures price converges to a fixed rate

$$\frac{1}{F}\frac{\partial F}{\partial T}(T \to \infty) = r + \lambda_S - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\rho\sigma_1\sigma_2}{\kappa} \quad (49)$$

In the basic model of equation (2), the rate of change in the futures price is

$$\frac{1}{F}\frac{\partial F}{\partial T} = r + \lambda_S - c \quad (50)$$

If the constant convenience yield in the long-term model is defined as

$$c = \alpha - \frac{\sigma_2^2}{2\kappa^2} + \frac{\rho\sigma_1\sigma_2}{\kappa} \quad (51)$$

it will have the same rate of change in futures prices as the two-factor model. Besides, since the objective is to match the futures prices, we must begin with a spot price to give the futures prices in equation (44) when the convenience yield in equation (51) is used. This starting price is referred to in Schwartz (1998) as the shadow spot price Z given by

$$Z(S, \delta) = \lim_{T \to \infty} e^{-(r+\lambda_S-c)T} F(S, \delta, T) \quad (52)$$

or

$$Z(S, \delta) = S e^{\frac{(c-\delta)}{\kappa} - \frac{\sigma_2^2}{4\kappa^2}} \quad (53)$$

When the shadow spot price is used as a single state variable in a model with a constant convenience yield c from equation (51), the model will show futures prices $F(Z, T)$ close to $F(S, ä, T)$ for T greater than three years.

The dynamics of the shadow spot price are given by

$$\frac{dZ}{Z} = (r + ë_S - c)dt + ó_F(t)dz \quad (54)$$

where the volatility is given by equation (46). In this case, the futures price for the shadow spot price Z is

$$F(Z, T) = Ze^{(r+ë_S-c)T} \quad (55)$$

Applying Ito's Lemma allows to show that the volatility of futures returns is $ó_F(T)$. The value of contingent claims in this model must satisfy the following PDE

$$\frac{1}{2}ó_F^2(T)Z^2V_{ZZ} + (r + ë_S - c)ZV_Z - V_T - (r + ë_V)V = 0 \quad (56)$$

under the terminal boundary conditions.

Using a two-factor model to redefine a single state variable (the shadow spot price), the resulting one-factor model is very similar to the basic model. The main di.erence is volatility which is time dependent

$$í(T) = \int_0^T ó_F^2(t)dt \quad (57)$$

with a closed-form solution

$$í(T) = \left(ó_1^2 + \frac{ó_2^2}{ê^2} - \frac{2ñó_1ó_2}{ê}\right)T + \frac{ó_2^2(1 - e^{-2êT})}{2ê^3} + \frac{2ó_2\left(ó_1ñ - \frac{ó_2}{ê}\right)(1 - e^{-êT})}{ê^2} \quad (58)$$

The risk-neutral distribution of the shadow spot price is log-normal. Its mean is given by equation (55). Its variance is given by equation (58). This variance is similar to that of the spot price in the two-factor model.

D. Valuing European Options

The one and two-factor models give very similar results for long-term options

because of the their nearly equal means and variances.

The value of a European call for both models is given by

$$C(., T) = e^{-(r+\lambda s)T} c(F(., T), T) \quad (59)$$

where

$$c(F, T) = FN[d] - KN[d - \nu(T)] \quad (60)$$

where

$$d = \left[\frac{\ln \frac{F}{K}}{\nu(T)} + \frac{1}{2}\nu(T)\right] \quad (61)$$

$$\nu(T) = \left(\sigma_1^2 + \frac{\sigma_2^2}{\kappa^2} - \frac{2\rho\sigma_1\sigma_2}{\kappa}\right)T + \frac{\sigma_2^2(1 - e^{-2\kappa T})}{2\kappa^3} + 2\sigma_2\left(\sigma_1\rho - \frac{\sigma_2}{\kappa}\right)\frac{(1 - e^{-\kappa T})}{\kappa^2} \quad (58)$$

The European call price and the futures price are functions of the spot commodity price and the convenience yield for the two factor model.

The call price and the futures price are a function of the shadow spot price in the long-term model. The variance is given by equation (57).

5. Implementation and optimal exercise criteria for American options

Schwartz (1998) implemented the long-term model using the estimated parameters from the two-factor model in Schwartz (1997). The analysis in these papers can be extended without major di.culties to account for the e.ects of incomplete information.

Table 1 provides the parameters for copper and oil. Publicly futures prices are used for copper for the period 1988-1995. Enron provided oil forward curves for the period 1993-1996.

Table 1 : Parameter Values for Oil and Copper in the period 1988-1996 : results of the two-factor model estimated in Schwartz (1997)

	Copper	Oil
Period	7/29/88-6/13/95	1/15/93-5/16/96
Contracts	F1,F3,F5,F7,F9	Enron-Data
N - observ	347	163
μ	0.326 (0.110)	0.082 (0.120)
κ	1.156 (0.041)	1.187 (0.026)
α	0.248 (0.098)	0.090 (0.086)
σ_1	0.274 (0.012)	0.212 (0.011)
σ_2	0.280 (0.017)	0.187 (0.012)

ñ 0.818 (0.020) 0.845 (0.024)
ë 0.256 (0.114) 0.093 (0.101)
ñ 0.06 0.05
Standard errors are in parenthesis.
The terms F1, F2,.. correspond to futures contracts with di.erent maturities.
It is possible to use Equation (44) in a double grid search routine to estimate the state variables S and ä, which minimize the squared deviation between model and market prices. The term structure of futures prices implied by the two-factor model can be constructed using equation (44) and the estimated state variables S and ä.
Equation (53) is used to estimate the shadow spot price Z.
Equation (55) allows the estimation of the term structure of futures prices implied by the long-term model.
Copper and oil futures contracts reported in the Wall Street Journal for 3/31/97 are used.
The extracted information is used to value European copper calls from the two-factor model and the long-term model for a strike price of 1 dollar.
The two models provide very similar results when the maturity if higher than three years.
The main question is : How the simple long-term model and the twofactor model compare with respect to the optimal time to undertake a project i.e. the optimal exercise of American options.
The critical spot price above which it is optimal to invest in the two-factor model depends on the current instantaneous convenience yield. The critical spot price in the long-term model is one critical shadow price.
18
Example
Consider a copper mine that can produce one ounce of copper at the end of each year for 10 years. The initial investment K = 2 and the unit cost of production C = 0.40. Assume that investment is done for three years and production starts at the end of the fourth year.
The first step determines the net present value of the project and the second step values the option to invest.
The NPV once the investment has been made is

$$NPV = Ó_{T=4}^{13} e^{-(r+ë_S)T} F(., T) - C Ó_{T=4}^{13} e^{-rT} - K \quad (59)$$

The summation starts at time 4 as the production.
The option to invest and the computation of the critical copper price can be determined by solving numerically the PDE (48) for the two-factor model

$$\frac{1}{2} ó_1^2 S^2 V_{SS} + \frac{1}{2} ó_2^2 V_{ää} + ó_1 ó_2 ñ S V_{Sä} + (r+ë_S-ä) S V_S + [ê(\hat{á}-ä)] V_ä - V_T - (r+ë_V) V = 0$$

under the condition

NPV = $Ó_{13}$

$_{T=4}e_{-(r+ës)T}F(., T)$ - $CÓ_{13}$

$_{T=4}e_{-rT}$ - K (59)

with

F(S, ä, T) = S exp[-ä

1 - $e_{-êT}$

ê

+ A(T)]

where

A(T) = (r+$ë_S$-ˆá+

1 2

$ó_2$

2

$ê_2$ -

$ó_1ó_2ñ$

ê

)T+

1 4

$ó_2$

2

1 - $e_{-2êT}$

$ê_3$ +(ˆáê+$ó_1ó_2ñ$-

$ó_2$

$_2$ ê

)

1 - $e_{-2êT}$

$ê_3$

and where the risk-adjusted long-run mean of the convenience yield process is given by

ˆá = á -

ë ê

where ë stands for the market price of convenience yield risk.

Equation (56) is solved for the long-term model

1 2

$ó_2$

$_F$ (T)Z_2V_{ZZ} + (r + $ë_S$ - c)ZV_Z - V_T - (r + $ë_V$)V = 0 (56)

19

under the boundary condition :

NPV = $Ó_{13}$

$_{T=4}e_{-(r+ësT}F(., T)$ - $CÓ_{13}$

$_{T=4}e_{-rT}$ - K (59)

with :

F(Z, T) = $Ze_{(r+ës-c)T}$ (55)

Z(S, ä) = Se

(c-ä)

ê -

ó2

2

4ê2 (53)
The boundary condition (59) must be applied.
The value of the mine in the two-factor model depends on the spot price and the convenience yield.
The optimal shadow price (1.12) in the long-term model is similar to that obtained from the two factor model. Hence, when valuing projects where cash flows start a few years later, a simple one-factor model can give practically the same results as a two-factor model. A similar analysis can be applied in the presence of shadow costs of incomplete information.
Summary
The pricing and hedging of commodity derivatives and natural resource investments
depend heavily on the dynamics of the underlying commodity.
Schwartz (1997) proposed three models which account for the mean reverting nature of commodity prices. The analysis reveals the importance of mean reversion in evaluating projects using the real options approach. The standard DCF approach seems to induce investment too early when prices are low while the real options approach seems to induce investment too late when prices are too high. This result appears when mean reversion is neglected.
This analysis is extended to account for the e.ects of incomplete information.
Schwartz (1997) showed that a two-factor model for the stochastic behavior of commodity prices fitted quite well the term structure of futures prices and futures return volatility in the case of copper and gold. However, this model is di.cult to apply for the valuation of projects with multiple options.
Schwartz (1998) proposed a simple one-factor model which gives nearly similar results as the two-factor model. The constant convenience yield in the simple model depends on the parameters of the two-factor model. The inputs to the model are the prices of all current futures contracts. The simple model can be applied to the valuation of complex projects. This analysis
20
is extended to account for the e.ects of incomplete information. We are actually estimating the parameters of the models and conducting some simulations.
REFERENCES
• Bellalah, M. and B. Jacquillat, " Option Valuation With Information Costs : Theory and Tests.", Financial Review, [1995], 617-635.
• Bellalah M., "The valuation of futures and commodity options with information costs", Journal of Futures Markets, [1999 a], September
• Bellalah M., "Les biais des modeles d'options revisites", Revue Franaise de Gestion, Juin, [1999 b] pp 94-100
• Bellalah M., " Irreversibility, Sunk costs and Investment Under Uncertainty", R&D Management, [2001],
• Bellalah M., " A Reexamination of Corporate Risks Under Incomplete Information", International Journal of Finance and Economics, [2001],
• Merton, R., "A Simple Model of capital market equilibrium with Incomplete Information.", Journal of Finance 42, [1987], 483-510.

• Schwartz E., ”The Stochastic Behavior of Commodity Prices : Implications for Valuation and Hedging”, Journal of Finance, Vol LII, No 3, July, [1997] pp 923-972
• Schwartz E., ”Valuing Long-Term Commodity Assets”, Journal of Energy Finance & development”, Vol 3, No 2, [1998], pp 85-99