THE GENERALISED FALKNER-SKAN EQUATION FOR NON-NEWTONIAN POWER-LAW FLUIDS

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SUMMARY

The Falkner-Skan equation has been derived for power-law non-Newtonian fluids using a novel method for the determination of the dimensionless parameters. As a result of its simplicity, the approach presented in this paper is particularly suitable for use in undergraduate degree courses.

1. INTRODUCTION

As a consequence of the expansion of the processing industries which handle these materials the study of non-Newtonian flow has increased in importance rapidly over the last fifteen years. Industries in which non-Newtonian fluids will be encountered, include those dealing with the following[1] rubber, plastics and synthetic fibres, petroleum, soap and detergents, pharmaceuticals, biological fluids, atomic energy, cement, foods, paper pulp, paint, light and heavy chemicals, fermentation processes, oil field operations, ore processing and printing. Indeed, the range is so vast that, to quote Wilkinson[2] ". . . . one could easily visualize a much broader approach to fluid mechanics into which the Newtonian fluid fits as a comparatively inconspicuous special case".

If this is so, why is it therefore, that present day engineering undergraduate courses barely touch upon the subject of non-Newtonian flow? One reason for this is the complex equations needed to model these fluids; another reason is that derivation of the equations for flow is generally very complicated requiring considerable knowledge of and dexterity with mathematics. The level of mathematics needed is very often beyond that which can be taught in a normal three-year undergraduate programme.

For example, in recent years there has been growing interest in the solution of the class of problems involving the external flows of non-Newtonian fluids. The majority of the published literature has appeared since 1965. Much of this work has been concerned with the formulation of the mathematics from the viewpoint of similarity solutions. Work of this nature may be found in references 3,4,5,6,7 and 8.

In all of this work, the mathematical approach used, with respect to non-dimensionalising the relevant equations, is fairly complicated and consequently beyond the comprehension of the majority of engineering undergraduates.

In this paper, a much simpler approach is used and it should be much easier for students to follow. The choice of the power law as the mathematical model is dictated by its fairly wide range of applicability as well as its simplicity. However, the approach reported here can be used for any rheological model desired.

2. THE BOUNDARY-LAYER EQUATIONS

2.1. The Continuity equation

Assuming two-dimensional, incompressible flow, this is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2.1)$$

2.2 The Navier-Stokes equation

For two-dimensional flow, the x and y components of the equation of motion are:

$$\rho \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} - \left\{\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right\} + \rho g_x \ldots \ldots (2.2)$$

$$\rho \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} - \left\{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right\} + \rho g_y \ldots \ldots (2.3)$$

The inviscid, external velocity distribution along x, U(x), is superimposed on the viscous region as a consequence of the classical approximation of no pressure variation across the boundary layer. From this assumption the following results:

$$-\frac{\partial p}{\partial x} = U\frac{dU}{dx}$$

Equations (2.2) and (2.3) can be simplified by using Prandtl's boundary-layer hypotheses:

i.e. $\delta \ll x$

and by considering the order of magnitude of each of the terms in the equations. The following conclusions may be drawn for steady-state conditions:

u, $u\frac{\partial u}{\partial x}$, $v\frac{\partial u}{\partial y}$, $U\frac{dU}{dx}$ and $\frac{\partial \tau_{yx}}{\partial y}$ are $0\ (1)$

$\therefore\ \tau_{yx} = \tau_{xy}$ is $0\ (\delta)$

and v, $u\frac{\partial v}{\partial x}$, $\frac{\partial p}{\partial y}$, $\frac{\partial \tau_{xy}}{\partial x}$, ρg_y are $0\ (\delta)$

where $0\ (1)$ and $0\ (\delta)$ indicate orders of magnitude and $x = 0(1)$; $u = 0(1)$. Hence the Navier-Stokes equation reduces to:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} - \frac{1}{\rho}\cdot\frac{\partial \tau_{xy}}{\partial y} \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2.4)$$

2.3 The Shear stress relationship

The rheological model to be used is the Ostwald-de-Wael (power law) model, which in tensor form is (see for example Bird et al[9])

$$\tau = -K\left\{ \left|\left[\tfrac{1}{2}(\Delta : \Delta)\right]^{\frac{1}{2}}\right|^{n-1}\right\} \Delta \quad \ldots\ldots\ldots\ldots (2.5)$$

where Δ = rate of deformation tensor

$$\text{and } \Delta_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

In two dimensions, after neglecting all terms of magnitude δ, equation (2.5) reduces to:

$$\tau_{xy} = -K\left|\frac{\partial u}{\partial y}\right|^{n-1} \frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots (2.6)$$

Velocity overshoot is not possible in boundary-layer flows, hence negative velocity gradients cannot occur. Consequently, equation (2.6) may be written as:

$$\tau_{xy} = -K\left(\frac{\partial u}{\partial y}\right)^{n} \quad \ldots\ldots\ldots\ldots (2.7)$$

2.4. The complete boundary-layer equations

After introducing the relationship for τ_{xy} from equation (2.7) into equation (2.4), the boundary-layer equations may be rewritten as:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \ldots\ldots\ldots\ldots (2.1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \frac{K}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^{n} \quad \ldots\ldots\ldots\ldots (2.8)$$

The boundary conditions are:

$$u(x,o) = v(x,o) = 0$$

$$\lim_{y \to \infty} u(x,y) = U(x) \quad \ldots\ldots\ldots\ldots (2.9)$$

3. THE DIMENSIONLESS TRANSFORMATION PARAMETERS

In order to obtain a similarity solution, it is necessary to derive dimensionless groups which combine the dependent and independent variables. Thus one group (η) will be derived which combines x, y with u, ρ and K; this will be known as the position ratio. The other group, the stream function (ψ) combines u and v.

3.1. The stream function

The stream function will be defined in accordance with equation (2.1) by,

$$u = \frac{\partial \psi}{\partial y} \; ; \; v = -\frac{\partial \psi}{\partial x} \quad \ldots\ldots\ldots\ldots (3.1)$$

3.2. The position ratio

In order to find a suitable dimensionless group (η), consider the dimensions of equations (2.1) and (2.8)

from (2.1) $\frac{u}{x} \propto \frac{v}{y}$ .... (3.2)

where $\triangleq$ stands for 'dimensionally equal to':
it therefore follows that $\frac{u^2}{x} \triangleq \frac{uv}{y}$ . . . . (3.3)

From (2.8) $\frac{u^2}{x}$, $\frac{uv}{y} \triangleq \frac{U^2}{x}$, $\frac{Ku^n}{\rho y^{n+1}}$ . . . . (3.4)

Hence two dimensionless groups can be obtained viz:

$$\frac{u^2}{U^2} \triangleq 0 \quad \text{and} \quad \frac{\rho u^{2-n}}{Kx} \cdot y^{n+1} \triangleq 0$$

The first dimensionless group can be used to eliminate u^2 from the second group. Thus

$$\rho\frac{U^{2-n}}{Kx} \cdot y^{n+1} \triangleq 0$$

Therefore the dimensionless position ratio to be used can be defined by

$$\eta = y\left\{\frac{\rho U^{2-n}}{Kx}\right\}^{\frac{1}{n+1}} \quad \ldots\ldots\ldots\ldots (3.5)$$

3.3. Relationship between η and ψ

If the dimensionless velocity profile is a function of the position ratio, then

$$\frac{u}{U} = \phi(\eta)$$

But from equation (3.1)

$$u = \frac{\partial\psi}{\partial y}_x = U\phi(\eta)$$

However by the chain rule of partial differentiation:

$$\left(\frac{\partial\psi}{\partial y}\right)_x = \left(\frac{\partial\psi}{\partial\eta} \cdot \frac{\partial\eta}{\partial y}\right)_x = \left\{\frac{\rho U^{2-n}}{Kx}\right\}^{\frac{1}{n+1}} \cdot \left(\frac{\partial\psi}{\partial\eta}\right)_x$$

Hence $\psi = \left(\frac{yU}{\eta}\right) \cdot f(\eta)$ . . . . (3.6)

where $\phi(\eta) = f'(\eta)$ . . . . (3.7)

4. THE BOUNDARY-LAYER EQUATIONS IN TERMS OF $f(\eta)$

Each of the terms in equations (2.8) and (2.9) can be evaluated as functions of by considering the definitions of η and ψ and by applying the rule for partial differentiation.

Thus for example:

$$v = -\frac{\partial\psi}{\partial x} = -\frac{\partial}{\partial x}\left[\left\{\frac{\rho U^{2-n}}{Kx}\right\}^{-\frac{1}{n+1}} U f(\eta)\right]$$

$$= -\left(\frac{\rho}{K}\right)^{-\frac{1}{n+1}} \frac{\partial}{\partial x}\left[U^{\frac{2n-1}{n+1}} x^{\frac{1}{n+1}} \cdot f(\eta)\right]$$

$$v = -\left(\frac{\rho}{K}\right)^{-\frac{1}{n+1}}\left[f(\eta)\frac{\partial}{\partial x}\left(U^{\frac{2n-1}{n+1}} x^{\frac{1}{n+1}}\right) + U^{\frac{2n-1}{n+1}} \cdot x^{\frac{1}{n+1}} \cdot \frac{\partial f(\eta)}{\partial x}\right] \quad (4.1)$$

Now $\frac{\partial f'(\eta)}{\partial x} = \frac{\partial f'(\eta)}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = f''\frac{\partial \eta}{\partial x}$

$$= f''\frac{\eta}{(n+1)}\cdot\frac{1}{xU}\{(2-n)\,xU' - U\}$$

$$\therefore v = -\left(\frac{\rho}{K}\right)^{-\frac{1}{n+1}}\left[\frac{1}{(n+1)x}\cdot\left(\frac{x}{\rho U^{2}}\cdot n\right)^{\frac{1}{n+1}}\cdot\{U+(2n-1)xU'\}\,f'(\eta) + U^{\frac{2n-1}{n+1}}\cdot x^{\frac{1}{n+1}}\cdot\frac{\eta}{(n+1)xU}\{(2-n)xU' - U\}\,f'(\eta)\right] \quad \ldots\ldots (4.2)$$

On rearrangement, the following expression for v is obtained

$$v = \frac{1}{(n+1)x}\cdot\left(\frac{Kx}{\rho U^{2}}\cdot n\right)^{\frac{1}{n+1}}[\{U-(2-n)xU'\}\eta f'(\eta) - \{U+(2n-1)xU'\}f(\eta)] \quad (4.3)$$

The other terms in equation (2.8) may be similarly transformed to give

$$Uf'(\eta)\left[\frac{\eta}{(n+1)x}\{(2-n)xU'-U\}f''(\eta) + f'(\eta)U'\right] + \frac{1}{(n+1)}\cdot\frac{U}{x}[\{U-(2-n)xU'\}\eta f'(\eta) - \{U+(2n-1)xU'\}f'(\eta)]\,f''(\eta) = UU' + \frac{U^{2}}{x}n.f'''(\eta)[f''(\eta)]^{n-1} \quad \ldots\ldots (4.4)$$

which on simplification becomes

$$n(n+1)f'''(\eta)[f''(\eta)]^{n+1} + \{1+(2n-1)\frac{xU'}{U}\}f(\eta)\,f''(\eta) + \frac{xU'}{U}(n+1).(1-[f'(\eta)]^{2}) = 0 \quad (4.5)$$

5. FALKNER-SKAN EQUATION

In order that this equation may be solved by the similarity method, the following condition must hold:

$$xU' = mU$$

$$\text{i.e. } U(x) = Cx^{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.1)$$

With this condition, equation (4.5) may be written as

$$n(n+1)f'''(\eta)[f''(\eta)]^{n-1} + \{1+(2n-1)m\}f(\eta)f''(\eta) + m(n+1)(1-[f'(\eta)]^{2}) = 0 \quad (5.2)$$

Equation (5.2) is the Falkner-Skan equation for a power-law non-Newtonian fluid. To obtain the Newtonian Falkner-Skan equation, n=1 can be substituted to give,

$$2f'''(\eta) + (1+m)f(\eta)f''(\eta) + 2m(1-[f'(\eta)]^{2}) = 0 \quad \ldots\ldots\ldots\ldots (5.3)$$

where $\eta = y\sqrt{\frac{\rho U}{\mu x}}$

6. CONCLUSION

It can be seen that the Falkner-Skan equation for a power-law fluid can be obtained fairly easily and with a very limited mathematical knowledge.

The method of dimensional analysis used in this paper is particularly simple and does not even require a knowledge of the units of the various terms. All that is necessary is a knowledge of which dependent and independent variables should be combined and the fact that the dimensions of quantities on each side of an equation must be the same.

The transformation of the boundary-layer equations can then be accomplished by using the chain rule of partial differentiation.

The method presented in this paper therefore, is particularly suitable as a tool in engineering undergraduate teaching especially in the subject area of transport phenomena.

NOMENCLATURE

Symbol	Definition	First used in Equation
K	Constant in power-law model	(2.5)
n	power-law index	(2.5)
p	pressure	(2.2)
t	time	(2.2)
u	x-direction component of velocity	(2.1)
v	y-direction component of velocity	(2.1)
U	free-stream velocity	(2.4)
x, y	directions in orthogonal component system	(2.1)
δ	boundary-layer thickness	
η	position ratio defined by $y\left(\rho\frac{U^{2-n}}{Kx}\right)^{\frac{1}{n+1}}$	(3.5)
ψ	stream function	(3.1)
ρ	density of fluid	(2.2)
	nine-component stress tensor	(2.5)
τ	rate of deformation tensor	(2.5)
Δ		

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