SUBOPTIMAL INTEGRAL STATE FEEDBACK CONTROL OF FINAL VALUE PROBLEMS

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Summary

Suboptimal control schemes incorporating integral state feedback and time-invariant gain parameters have been suggested for dynamic optimisation problems without control constraints. Necessary conditions have been obtained in respect of a general suboptimal control configuration and specifically applied to linear and nonlinear final value control problems. The suboptimal schemes which can handle a variety of control situations also appear to be good from the point of view of feasibility of synthesis and ease of implementation on-line. It is felt that the suboptimal feedback structure proposed will be useful for many practical systems that are otherwise difficult to be optimally controlled in a feedback fashion.

1. INTRODUCTION

Modern control system studies and applications essentially demand 'best' system performance on the basis of a chosen objective function and the implementation of an optimal control policy to achieve the same. The optimization is carried out in the time domain by the state variable approach extremising a desired performance criterion function. Calculus of variations, Pontriagin's Maximum Principle and Bellman's Dynamic Programming give the necessary mathematical background for the dynamic optimization problem. It may be true that the implementation of the optimal control is seldom feasible practically, but still, the knowledge of the same is desirable at least for comparison of the actual performance with the 'ideal' one.

Perhaps, the most widely studied class of optimal control problems is the linear quadratic problem. Here the system dynamics is linear and the performance functional is a quadratic function of the state and control. The optimal control law is linear and can be obtained explicitly. The optimal gain is a time-varying function, independent of the initial conditions, and may be obtained by solving the matrix riccati equations. Although these results are elegant and appealing, the on-line implementation of the optimal control can still be difficult, especially for higher order systems. Possibly then, one may have to go in for something less than optimal — a suboptimal control law.

For non-linear systems and/or nonquadratic costs, the explicit realization of the optimal control as a closed loop policy is itself extremely difficult, if not almost impossible. This is because, Pontriagin's Minimum Principle yields essentially an open loop control function and the Dynamic Programming approach leads to the Hamilton Jacobi-Bellman partial differential equation whose exact solution is seldom possible in almost every non-linear case [1]. Thus, since the advantage of a feedback control law over an arbitrary control function is well known, the alternative is a suboptimal feedback control policy.

A suboptimal control, very often, is optimal subject to the additional simplifications and constraints imposed on to synthesize the same. For reasons already mentioned, controls other than truly optimal controls are employed and this is bound to result in a corresponding performance degradation, the percentage change in the performance index value being considered as one measure of the degree of suboptimality. Such a suboptimality may be due to any one of the following reasons or other considerations:

(i) Constraints are imposed on the control structure from the point of view of feasibility of synthesis and ease of implementation.
(ii) The truly optimal control may be solved for, in terms of an approximate or feasible policy.
(iii) The original system dynamics may be simplified before an optimal solution is obtained.

All these controls are referred to as suboptimal controls in the literature. When the control is dependent on the present state directly or indirectly and when it can be synthesized in a feedback fashion, the suboptimal control may be called a suboptimal policy or control law.

Suboptimal control using piecewise-constant feedback gains, constant output feedback gains, integral feedback for the control, etc., have been studied in the literature especially for linear systems. Yet another suboptimal approach seems to be the specific optimal control [2], where the control configuration is predetermined and the problem reduced to one of parameter optimisation. Suitable control configurations, essentially of a feedback nature, are proposed and the unknown time-invariant parameters are selected optimally to minimize the given cost. The significant contribution is the introduction of integral state feedback and its application to treat a variety of control situations amenable to such an approach. A class of suboptimal feedback policies for linear and nonlinear systems with and without incorporating integral state feedback is stuided in [3]. A proportional plus proportional-integral suboptimal control scheme suitable for nonlinear systems and non-quadratic costs has been given in [4]. In the present paper, necessary conditions are obtained for a general suboptimal control scheme incorporating integral state feedback and time-invariant feedback gains for dynamic optimisation problems. The results are then applied to linear and non linear terminal control problems.

2. SUBOPTIMAL CONTROL SCHEMES

Consider the dynamic system modelled by the nonlinear vector differential equation:

$$\dot{x} = f(x,u,t); \quad x(t_o) = x_o \tag{2-1}$$

where x is the state vector in Euclidean n space E^n, u is the control vector in E^m, t is the scalar representing time and a dot over a symbol denotes differentiation with respect to time. The instants t_o and t_f are fixed and known and f (.) is a given vector function of x, u and t. The initial state vector x_o is available deterministically and, without loss of generality, $x(t_f)$ is assumed free.

The ideal goal is to minimize the scalar performance index:

$$J = \theta(x(t_f)) + \int_{t_o}^{t_f} \phi(x,u,t)\,dt \tag{2-2}$$

with respect to the control u, which is unconstrained except for that it must be continuous in time. θ (.) and ϕ (.) are scalar functions of their respective arguments.

A sufficiently general and useful suboptimal control format incorporating integral feedback, arbitrary time functions and time-invariant gains may be conveniently specified as:

$$u(t) = [\sum_{i=1}^{M} L_i(t)\, A^i\,]\, x(t) + [\sum_{j=1}^{N} \beta_j(t)\, B^j\,] \int_{t_o}^{t} x(t)\,dt \tag{2-3}$$

where A^i, $i = 1, \ldots, M$ and B^j, $j=1, \ldots, N$ are mxn constant matrices; $L_i(t)$ and $B_j(t)$ are scalar functions of time to one's choice; and M and N are integers. The scalar functions, although arbitrary, should be chosen judiciously so that they are simple and easily implementable on-line. It may also be noted that the control scheme (2-3) is of a proportional plus integral configuration. Now, the necessary conditions to be satisfied by the optimal A^i and B^j are obtained using the variational approach.

For convenience, let A^i and B^j be partitioned into n columns each:

$$A^i = [\, a_1^i \quad a_2^i \quad \ldots \quad a_n^i \,]$$

$$B^j = [\, b_1^j \quad b_2^j \quad \ldots \quad b_n^j \,]$$

Here, $a_1^i, a_2^i, \ldots, a_n^i$ and $b_1^j, b_2^j, \ldots, b_n^j$ are all m-vectors. The requirements that the elements of A^i and B^j are constants imply that :

$$\dot{a}_1^i = 0; \dot{a}_2^i = 0, \ldots . \dot{a}_n^i = 0; \ i = 1, \ldots, M \tag{2-4a}$$

$$\dot{b}_1^j = 0, \dot{b}_2^j = 0, \ldots, \dot{b}_n^j = 0; \ j = 1, \ldots, N \tag{2-4b}$$

Define the relation,

$$\int_{t_0}^{t} x(t)\,dt = y(t) \tag{2-5}$$

This means that :

$$\dot{y} = x; \ y'(t_0) = 0 \tag{2-6}$$

Thus $y(t)$ is also a vector in E^n like the state $x(t)$. Now, the augmented performance index after suppressing the arguments may be written as :

$$J_a = \theta(x(t_f)) + \int_{t_0}^{t_f} [\phi + \lambda'(f - \dot{x}) + \xi'(x - \dot{y}) - \sum_{i=1}^{M} \sum_{k=1}^{n} \eta_k'^i \dot{a}_k^i - \sum_{j=1}^{N} \sum_{l=1}^{n} \gamma_l'^j \dot{b}_l^j] \, dt \tag{2-7}$$

where λ, ξ, $\eta_k^i, i = 1, \ldots, M, k=1, \ldots, n$ and $\gamma_l^j, j = 1, \ldots, N, l = 1, \ldots, n$ are appropriate vector lagrange multipliers to account for the constraints (2-1), (2-6), (2-4a) and (2-4b) respectively. The Hamiltonian as usual, is :

$$H = \phi + \lambda' f + \xi' x \tag{2-8}$$

δJ_a the first variation of J_a can be easily shown to be :

$$\delta J_a = \left[\frac{\partial \theta(x(t_f))}{\partial x(t_f)} - \lambda(t_f) \right]' \delta x(t_f) - \xi'(t_f)\, \delta y(t_f) + \sum_{i=1}^{M} \sum_{k=1}^{n} \left[\eta_k^i(t_o) - \eta_k^i(t_f) \right]' \delta a_k^i + \sum_{j=1}^{N} \sum_{l=1}^{n} \left[\gamma_l^j(t_o) - \gamma_l^j(t_f) \right]' \delta b_l^j + \int_{t_o}^{t_f} \Bigg\{ \left(\frac{\partial H}{\partial x} + \dot{\lambda} \right)' \delta x + \left(\frac{\partial H}{\partial y} + \dot{\xi} \right)' \delta y + \sum_{i=1}^{M} \sum_{k=1}^{n} \left(\frac{\partial H}{\partial a_k^i} + \dot{\eta}_k^i \right)' \partial a_k^i +$$

$$\sum_{j=1}^{N} \sum_{l=1}^{n} \left(\frac{\partial H}{\partial b_l^j} + \dot{\gamma}_l^j \right)' \delta b_l^j \Bigg\} dt \qquad (2\text{-}9)$$

δJ_a is a zero when :

$$\frac{\partial H}{\partial x} = -\dot{\lambda} \; ; \; \lambda(t_f) = \frac{\partial \theta(x(t_f))}{\partial x(t_f)} \qquad (2\text{-}10)$$

$$\frac{\partial H}{\partial y} = -\dot{\xi} \; ; \; \xi(t_f) = 0 \qquad (2\text{-}11)$$

$$\frac{\partial H}{\partial a_k^i} = -\dot{\eta}_k^i \; ; \quad i = 1, \ldots, M; \; k = 1, \ldots, n \qquad (2\text{-}12a)$$

$$\eta_k^i(t_o) = \eta_k^i(t_f); \quad i = 1, \ldots, M; \; k = 1, \ldots, n \qquad (2\text{-}12b)$$

$$\frac{\partial H}{\partial b_l^j} = -\dot{\gamma}_l^i \; ; \quad j = 1, \ldots, N; \; l = 1, \ldots, n \qquad (2\text{-}13a)$$

$$\gamma_l^j(t_o) = \gamma_l^j(t_f); \quad j = 1, \ldots, N; \; l = 1, \ldots, n \qquad (2\text{-}13b)$$

Conditions (2-12) and (2-13) are respectively equivalent to :

$$\int_{t_0}^{t_f} \frac{\partial H}{\partial a_k^i} dt = 0; \quad i = 1, \ldots, M; \; k = 1, \ldots, n \qquad (2-14)$$

$$\int_{t_0}^{t_f} \frac{\partial H}{\partial b_l^j} dt = 0; \quad i = 1, \ldots, N; \; l = 1, \ldots, n \qquad (2-15)$$

Thus the necessary conditions of optimality may be consolidated and given as :

$$\dot{x} = f\left\{ x, \; [(\sum_{i=1}^{M} L_i(t) \, A^i) \, x + (\sum_{j=1}^{N} \beta_j(t) \, B^j) y], \; t \right\}; \; x(t_0) = x_0 \qquad (2-16)$$

$$\dot{y} = x; \qquad y(t_0) = 0 \qquad (2-17)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x}; \qquad \lambda(t_f) = \frac{\partial \theta \, (x(t_f))}{\partial \, x(t_f)} \qquad (2-18)$$

$$\dot{\xi} = -\frac{\partial H}{\partial y}; \qquad \xi(t_f) = 0 \qquad (2-19)$$

$$\int_{t_0}^{t_f} \frac{\partial H}{\partial A^i} dt = 0; \quad i = 1, \ldots, M \qquad (2-20)$$

$$\int_{t_0}^{t_f} \frac{\partial H}{\partial B^j} dt = 0; \quad j = 1, \ldots, N \qquad (2-21)$$

Basic results being the same, a number of different suboptimal control schemes may be formulated with and without integral feedback as particular cases of the general suboptimal control structure. In the following sections, a few illustrative control schemes are considered for comparison and discussion to solve linear and non-linear terminal control problems.

3. FINAL VALUE CONTROL PROBLEMS

Terminal control problems or final value control problems are optimal problems which have to satisfy exactly all or some of the terminal conditions which are specified [7]. Formulating steering laws for missile systems or designing guidance policies for space rendezvous and docking missions normally fall into this category of control problems. A linear quadratic problem, with specified terminal states to be satisfied exactly at the final time, has the optimal closed loop gain tending to become infinite as the terminal time is approached [5], [6], even though the control remains finite. Feedback of the states alone cannot possibly meet the terminal constraints and suitable modifications of the basic result may be required to treat such Final Value Problems [7].

In this context, the following two general suboptimal control schemes are suggested to treat final value control problems.

$$u(t) = [\sum_{i=1}^{M} L_i(t) A^i]\, x(t) + \sum_{j=1}^{M} \beta_j(t)\, b^j \tag{3-1}$$

$$u(t) = [\sum_{i=1}^{M} L_i(t) A^i]\, x(t) + [\sum_{j=1}^{N} \beta_j(t) B^j] \int_{t_o}^{t} x(t)\,dt \tag{3-2}$$

where $L_i(t)$ and $\beta_j(t)$ are known scalar time functions; A^i, b^j, β^j represent appropriate constant gain parameters to be selected optimally and M and N are integers. Note that (3-2) is the same as (2-3) introduced in Section 2 for which the necessary conditions are obtained, whereas, (3-1) is its reduced form without incorporating integral state feedback. The control structures are chosen such that they may be synthesized in a feedback fashion from the present state, present time and integral of the state and combinations thereof with time-invariant gains from the point of view of easy implementation. Simple and illustrative versions of the above control schemes are applied to scalar examples of final value control problems.

The necessary conditions for optimal parameter selection in respect of each of the above suboptimal control schemes are obvious from the discussions in Section 2. Note that for a terminal control problem, the dynamic system (2-1) has not only initial state vector specified but also terminal conditions $x(t_f) = x_f$ to be satisfied exactly at the specified terminal time t_f. Obviously, the terminal state penalty function $\theta(x(t_f))$ is absent in the performance measure (2-2). Consequently, the terminal conditions on the corresponding costate in (2-18) are absent but replaced by the state terminal constraints $x(t_f) = x_f$. In fact, neither initial nor final conditions can be imposed on the costate $\lambda(t)$, since the state vector is specified both at the initial and final time t_o and t_f respectively.

4. COMPUTATIONAL ASPECTS

From the necessary conditions (2-16) through (2-21) it is clear that for an n^{th} order dynamic system with r design parameters of the suboptimal controller, a suboptimal solution is obtained when $4n$ first ordered differential equations with split boundary conditions are solved simultaneously satisfying r integral conditions over the control interval. Thus, any computational algorithm has the following aspects:

(i) Solution of the differential equations for known (assumed) values of the parameters.

(ii) An iterative procedure leading to the optimal choice of the parameters which satisfy the integral conditions.

Although the differential equations have split boundary conditions, the solution of the two-point boundary value problem is not difficult since the state equations are uncoupled with the costate variables. Hence, the state equations may be integrated in the forward time direction with known initial conditions to obtain the values at the terminal time. With these terminal conditions, and the known (assumed) terminal conditions of the costate variables, all the equations are integrated backward in time, simultaneously evaluating the integral expressions corresponding to each parameter value, which should vanish independently for the solution sought. Fourth-order Runge-Kutta integration procedure with an appropriate, step size, and evaluation of the definite integral by Simpson's Rule are normally satisfactory. A modified Newton-Raphson iterative procedure may be used to satisfy the integral conditions and converges fast provided the initial guess is reasonably good. When the number of unknown quantities is large, especially with poor initial values, it is suggested that a minimisation procedure such as Powell's may be advantageously employed to satisfy the integral conditions, although this may take more computer time.

In the case of final value problems, as the state vector is specified at either boundaries, the computational procedure requires modification since neither the initial nor the final value of the corresponding costate is known. The iterative procedure has not only to satisfy the integral conditions in respect of the constant gains to be determined, but also to choose the particular costate to meet the terminal constraints on the state. Linear and nonlinear scalar final value problems are now worked out for numerical results.

5. EXAMPLES

5.1. Example

Consider the scalar linear quadratic final value problem:

$$\dot{x} = -x + u; \qquad t \in [0,1]$$

$$x(0) = 1; \; x(1) = x_f$$

$$J = \tfrac{1}{2}\int_0^1 (x^2 + u^2)\,dt$$

The optimal closed loop control, in this case can be shown to be:

$$u^* = \left[\frac{2\sqrt{2}}{\exp(\sqrt{2}(1-t)) - \exp(-\sqrt{2}(1-t))}\right] x_f -$$

$$\left[\frac{(\sqrt{2}+1)\exp(\sqrt{2}(1-t)) + (\sqrt{2}-1)\exp(-\sqrt{2}(1-t))}{\exp(\sqrt{2}(1-t)) - \exp(-\sqrt{2}(1-t))}\right] x(t)$$

As $t \to 1, \; u^* \to \dfrac{x_f - x(t)}{1 - t}$

The singularity structure of the truly optimal closed loop control at the final time is clear from the above result. In this context, the following simple suboptimal controls are considered for illustration and comparison.

(i) $u(t) = ax(t) + b$

(ii) $u(t) = ax(t) + bt$

(iii) $u(t) = ax(t) + b \int_0^t \dot{x}(t)\,dt$

(i) is obtained with $M = N = 1;\; L_i(t) = \beta_j(t) = 1$ in (3-1)

(ii) is obtained with $M = N = 1;\; L_i(t) = 1; \beta_j(t) = t$ in (3-1)

(iii) is obtained with $M = N = 1;\; L_i(t) = \beta_j(t) = 1$ in (3-2)

Also, (i) is a proportional control with a bias, (ii) is a proportional control supplemented with an easily generated ramp function on a (iii) is a proportional plus integral control. Each one may be easily generated and implemented on-line. It is true that more complex control configurations can always be formulated similarly from the general control structures given, possibly with nominal advantages in certain cases, but normally only at the expense of simplicity of synthesis and implementation. The necessary conditions are obtained and the problem is solved on a digital computer in respect of each of the above control schemes for a desired terminal state $x_f = 0$ and 0.2. The optimal parameter gains and the performance index value is for all cases are given in Table I. The truly optimal controls are also worked out and performance index values given for comparison. The control and state trajectories are given in Figures 1, 2 & 3. The optimal state trajectories for $x_f = 0$ and 0.2 are extremely close to the respective firm curves in Figure 3 and hence are not shown separately.

TABLE I

NUMERICAL RESULTS FOR EXAMPLE 5.1

CONTROL	$x_f = 0$			$x_f = 0.2$		
	a	b	J	a	b	J
Scheme (i)	0.1257	-0.6257	0.296971	-0.3267	-0.1221	0.201687
Scheme (ii)	-0.5533	-0.6673	0.296231	-0.4413	-0.1121	0.201625
Scheme (iii)	-0.4433	-1.3108	0.298893	-0.4262	-0.1996	0.201679
Optimal Control	—	—	0.295944	—	—	0.201616

On the basis of these results, it is seen that the suboptimal control schemes considered here compare quite favourably with the truly optimal case. The corresponding performance index values also lie within one percent compared to the optimal one. Suboptimal control incorporating integral state feedback has been shown to handle non-linear optimal problems with terminal state constraints in the next example.

5.2 Example

Consider the first order nonlinear final value problem:

$$\dot{x} = -x^3 + u; \quad x(o) = 1, \; x(1) = 0, \; t \in [0,1]$$

A quadratic performance index to be minimised is:

$$J = \frac{1}{2} \int_0^1 (x^2 + u^2)\, dt$$

Assume a proportional plus integral suboptimal control of the form:

$$u(t) = ax(t) + b \int^t x(t)\, dt$$

where a and b are scalar constant gains to be chosen in an optimal fashion. The following necessary conditions which are to be satisfied may be easily written down.

$$\dot{x} = -x^3 + ax + by \; ; \qquad x(o) = 1.0$$

$$\dot{y} = x; \qquad y(o) = 0.0$$

$$\dot{\lambda} = -((a^2+1)\, x + aby - 3\lambda x^2 + a\lambda + \xi); \quad x(1) = 0.0$$

$$\dot{\xi} = -(abx + b^2 y + b\lambda); \qquad \xi(1) = 0.0$$

$$\int_0^1 x\,(ax + by + \lambda)\, dt = 0$$

$$\int_0^1 y\,(ax + by + \lambda)\, dt = 0$$

One may note that neither the initial nor the final value of λ is known since x is specified at both ends. Thus the iterative procedure should, in fact, select not only a and b but also the particular value of λ. The solution on a digital computer gives the suboptimal control as:

$$u(t) = -0.6419\, x(t) - 1.6974 \int_0^t x(t)\, dt$$

The performance index value 0.461726 as against the truly optimal open loop value of 0.461560. Figure 5.4 gives the controls and the suboptimal state trajectory.

It is seen that the optimal state trajectory is very close to the suboptimal one and hence it is not shown in Figure 4. The performance degradation is <0.04%. The suboptimal scheme has the added advantage of easy implementation on account of the desirable feature of an essentially feedback configuration.

6. CONCLUSION

Suboptimal control with and without integral state feedback and optimally selected constant gains, is discussed for dynamic optimisation problems. Illustrative examples of linear and nonlinear final value problems are worked out for numerical results. It is concluded that a proportional plus integral control configuration greatly improves system optimisation even under state terminal constraints.

The optimal parameters may be computed off-line and since the integrals of the states can be easily generated on line, it appears that the suboptimal feedback policy is also suitable for on-line implementation. Modifications, if required, when certain states are not accessible, may be incorporated easily. The dependence of the optimal parameters on the initial conditions is a limitation in the linear case. However, it may be possible to obtain parameters independent of the initial conditions, although with a further degration in performance, by treating the initial states as a random vector and minimising the expected value of performance measure. This aspect is not considered in the present work.

7. REFERENCES

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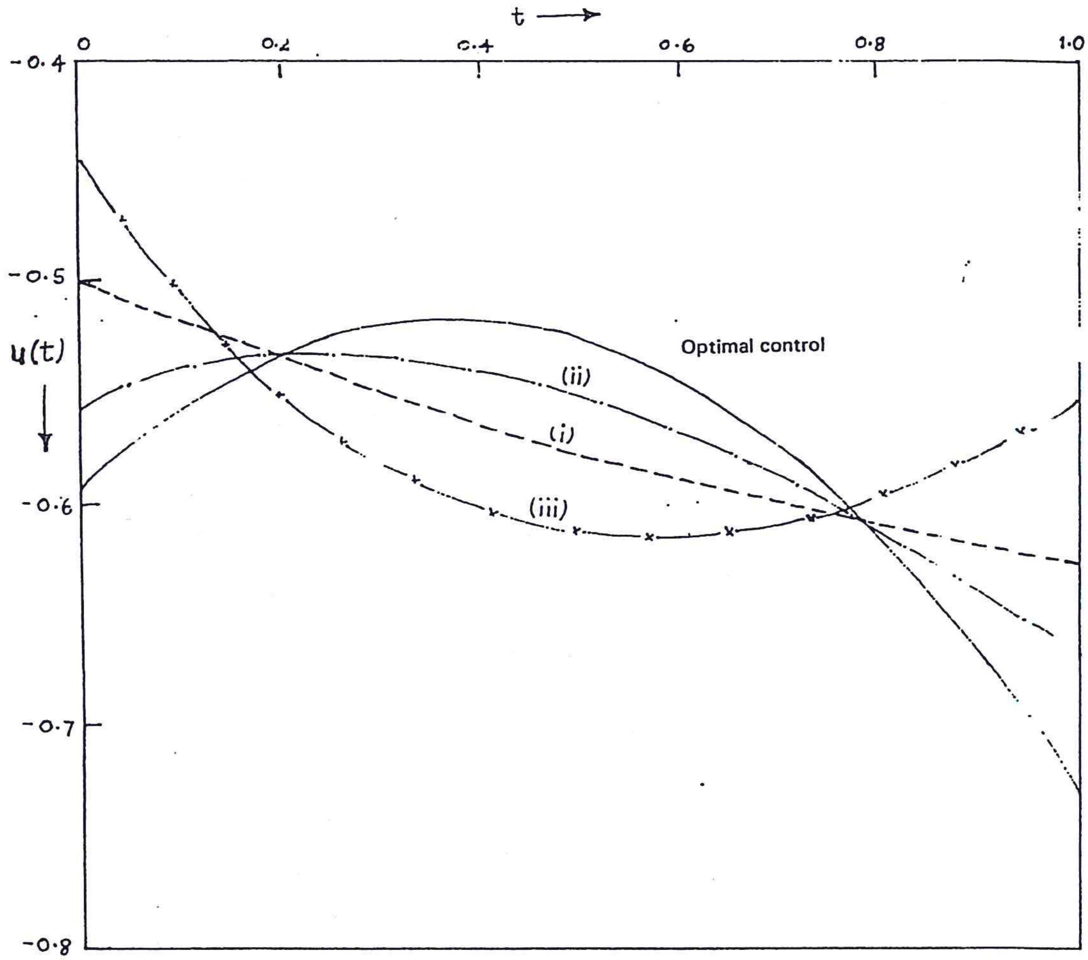

Fig. 1: Controls against time for Example 5.1 $x_f = 0.0$

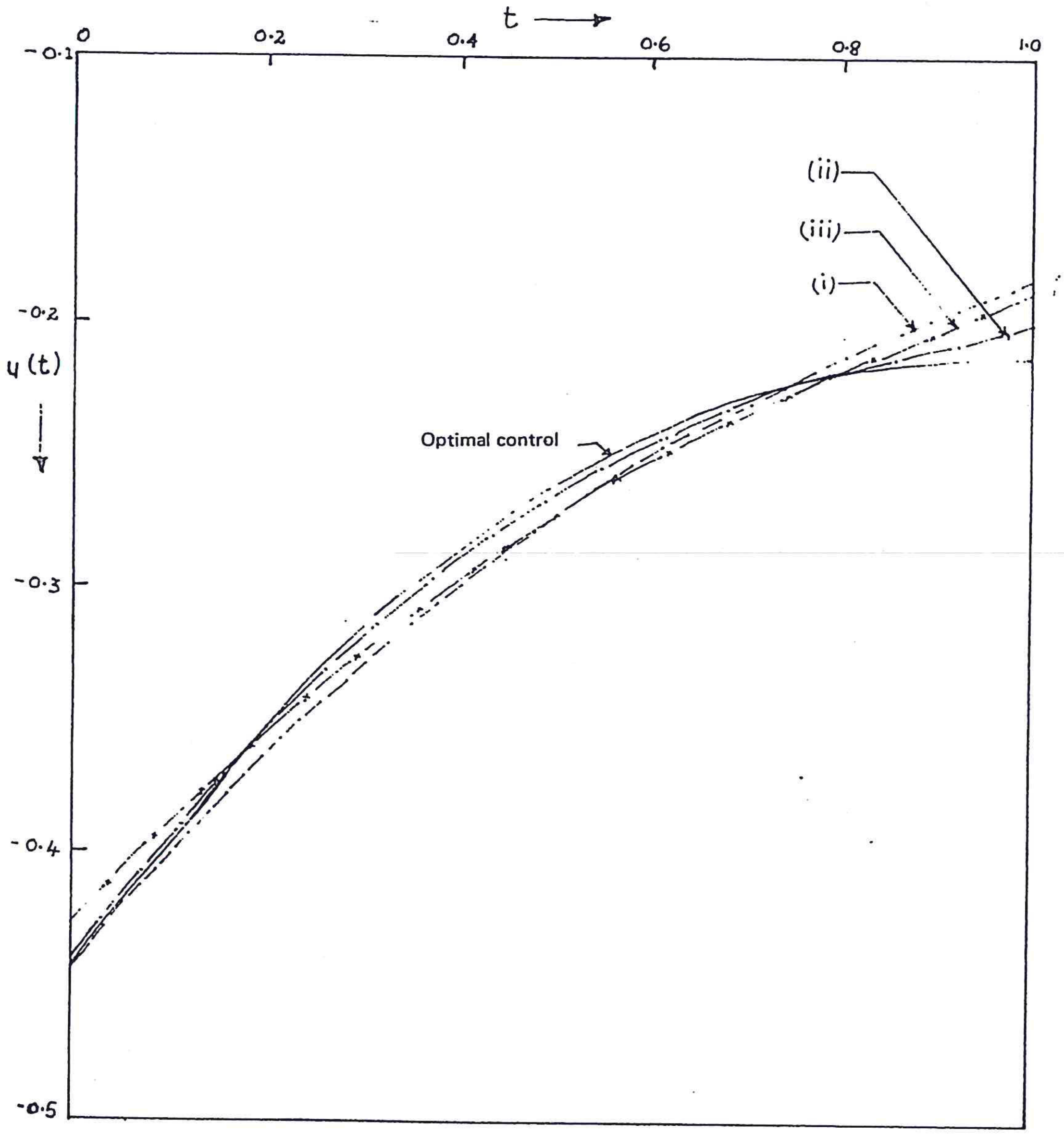

Fig. 2: Controls against time for Example 5.1, $x_f = 0.2$

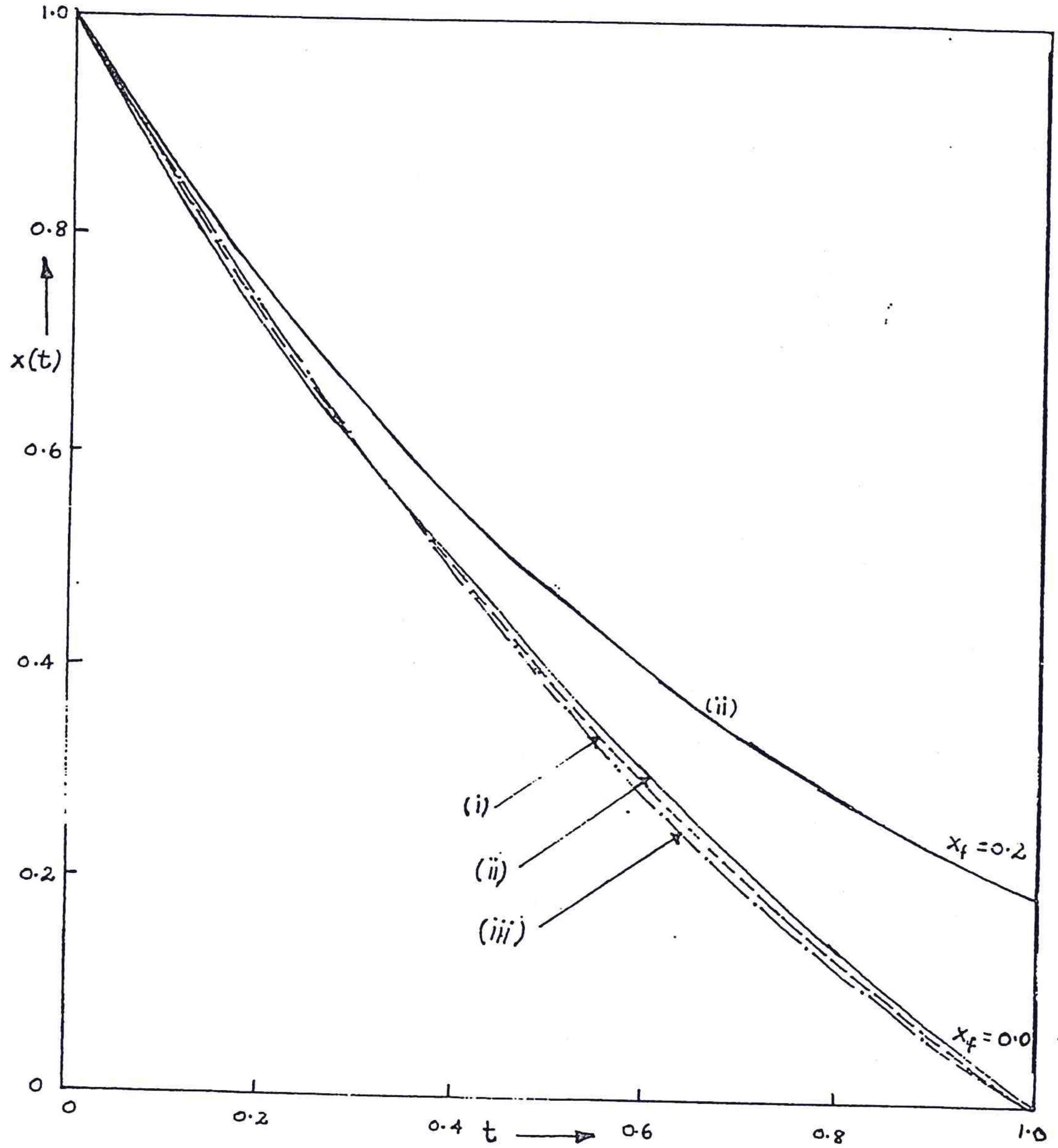

Fig. 3: State trajectories for Example 5.1

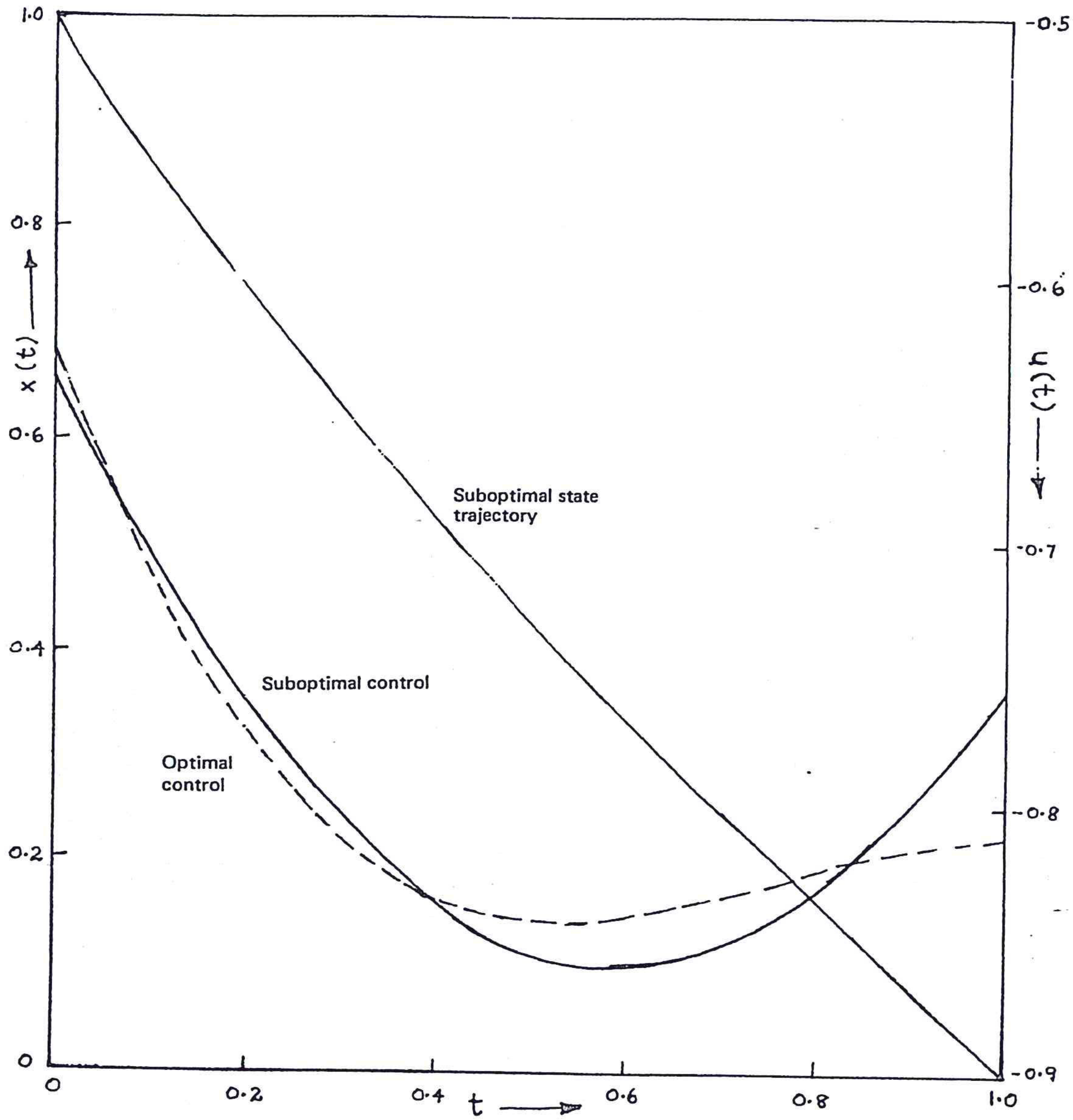

Fig. 4: Control & state trajectories for Example 5.2