THE DISPLACEMENT FIELD GENERATED BY TRACTIONS APPLIED TO THE SURFACE OF A CYLINDRICAL CAVITY IN AN ELASTIC MEDIUM

B. Alabi
Mechanical Engineering Department,
University of Ibadan,
Ibadan, Nigeria.

Summary

A numerical method is presented for the evaluation of the displacement field generated in an infinite elastic medium as a result of time-harmonic tractions applied to the surface of an embedded infinite cylindrical cavity whose cross-section has an arbitrary shape. Mapping is used to condense the physical problem domain. Transformed field equations are expressed in finite-difference forms and solved in a rectangular region.

1. INTRODUCTION

Most of the practical problems which arise in Engineering exhibit certain essentail features which preclude exact analytical solutions. Some of these features are awkward boundary shapes and non-linear boundary conditions at known or in some cases unknown boundaries. Thus, it is often necessary to resort to approximations, numerical sôlutions or combinations of both.

The advent of fast and large digital computers, has led to recent advances in the development of a variety of numerical methods of solution to these problems. However, well known numerical methods such as the Finite Difference method, Finite Element method and the method of Least Squares, pose serious and often insurmountable difficulties when they are implemented for the solution of problems with domains having awkward boundaries or when the problem domain is of an infinite extent.

The purpose of this paper is to present a numerical method which can be applied advantageously to boundary-value problems in the theory of time-harmonic elastic vibrations involving infinite domains and complex boundary shapes. In particular, the numerical evaluation of the displacement field generated in an infinite elastic medium, as a result of time-harmonic tractions applied to the surface of an embedded infinite cylindrical cavity whose cross-section has an arbitrary shape is described. The numerical procedure is a blend of the Conformal Mapping technique and the Finite Difference method; with the mapping being used primarily to condense the physical problem domain. Numerical results are presented and compared with exact analytical solutions for an illustrative problem.

2. FORMULATION

Let X_i; i = 1,3 denote the coordinates of a point in a rectangular Cartesian coordinate system. Consider a vector displacement field $\hat{U}\ (X_1, X_2)e^{i\omega t}$ in an infinite elastic medium which is also assumed to be isotropic, homogenous and unlayered. $\hat{U}$ represents a train of waves propagating through the medium as a result of the application of time-harmonic tractions to the surface of an infinite cylindrical cavity embedded in the elastic medium. The infinite length of the cylinder runs along the X_3-axis, while its cross-section which is assumed to be of an arbitrary shape lies in the X X_2-plane.

In the X_i coordinate system, the equations of motion for the elastic medium are

$$c_1^2\left(\frac{\partial^2 U}{\partial X_1^2} + \frac{\partial^2 V}{\partial X_1 \partial X_2}\right) + c_2^2\left(\frac{\partial^2 U}{\partial X_2^2} - \frac{\partial^2 V}{\partial X_1 \partial X_2}\right) = \frac{\partial^2 U}{\partial t^2} \qquad (1)$$

$$c_1^2\left(\frac{\partial^2 V}{\partial X_2^2} + \frac{\partial^2 U}{\partial X_1 \partial X_2}\right) + c_2^2\left(\frac{\partial^2 V}{\partial X_1^2} - \frac{\partial^2 U}{\partial X_1 \partial X_2}\right) = \frac{\partial^2 V}{\partial t^2} \qquad (2)$$

where the physical components of displacements, U_i, are denoted by $U_1 = U$ and $U_2 = V$, c_1 is the propagation velocity of dilatational waves, and c_2 the propagation velocity of distortional waves given as

$$c_1^2 = (\lambda + 2G)/\rho$$

$$c_2^2 = (\lambda + 2G)/\rho$$

in which λ and G are Lamé constants and ρ is the mass density of the medium.

In an orthogonal coordinate system equations (1) and (2) may be written as

$$c_1^2 \frac{\partial}{\partial x_1}[(1/\sqrt{g_{11}g_{22}})(\frac{\partial}{\partial x_1}(u\sqrt{g_{22}/g_{11}}) + \frac{\partial}{\partial x_2}(v\sqrt{g_{11}/g_{22}}))]$$

$$-c_2^2 \sqrt{g_{11}/g_{22}} \frac{\partial}{\partial x_2} [(1/\sqrt{g_{11}g_{22}})(\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2})] = \frac{\partial^2 u}{\partial t^2} \qquad (3)$$

$$c_1^2 \frac{\partial}{\partial x_2}[(1/\sqrt{g_{11}g_{22}})(\frac{\partial}{\partial x_1}(u\sqrt{g_{22}/g_{11}}) + \frac{\partial}{\partial x_2}(v\sqrt{g_{11}/g_{22}}))]$$

$$+c_2^2 \sqrt{g_{22}/g_{11}} \frac{\partial}{\partial x_1} [(1/\sqrt{g_{11}g_{22}})(\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2})] = \frac{\partial^2 v}{\partial t^2} \qquad (4)$$

where u_i the covariant components of the displacement vector on the reciprocal reference base vector $\hat{g}^i$ of the x_i orthogonal curvilinear coordinates are denoted by $u_1 = u$ and $u_2 = v$, while

$$g_{mn} = (\partial X_k/\partial x_m)(\partial X_k/\partial x_n)$$

are the components of the metric tensor associated with the x_i coordinate system defined by

$$X_i = X_i(x_1, x_2) \qquad i = 1,2 \qquad (6)$$

In two dimensions, orthogonal coordinates with specified coordinate lines are conveniently found through conformal mapping. Consider a conformal mapping which carries the arbitrary cross-section of the cylindrical cavity in the physical coordinates X_i into the unit circle in the x_i coordinates and the region exterior to the cross-section of the cylinder into the region exterior to the unit circle. In terms of complex variable notation, this may be expressed as

$$Z = (b_1 + id_1)z + \sum_{n=2}^{N} (b_n + id_n)z^{1-n} \tag{7}$$

where

$$Z = X_1 + iX_2 \tag{8}$$

and

$$z = x_1 e^{ix_2} \tag{9}$$

For cases in which the cross-section of the cylindrical cavity is symmetric about the X_1 and X_2 axes, n may be only even and $d_n = 0$. The transformation equation (7) then becomes

$$Z = \sum_{n=1}^{N} a_n z^{3-2n} \tag{10}$$

For (10) the Cauchy-Riemann equation imply that $g_{12} = 0$, and therefore that x_i is an orthogonal coordinate system. It also follows that

$$g_{22} = x_1^2 g_{11} \tag{11}$$

An effective numerical method for the computation of the coefficients a_n, occurring in the mapping function (10) is given in [1] and may be described as follows: Equating real and imaginary parts of (10) and noting that on the unit circle $z = e^{ix_2}$, gives

$$X_{1j} = \sum_{n=1}^{N} a_n \cos(3-2n)x_{2j} \tag{12}$$

$$X_{2j} = \sum_{n=1}^{N} a_n \sin(3-2n)x_{2j} \tag{13}$$

where X_{1j} and X_{2j} are the coordinates of some point on the boundary of the cross-section of the cylindrical cavity in the physical Z-plane, and x_{2j} is the coordinate of the corresponding point on the unit circle in the mapping z-plane. Let $j = 1,M$. If $N = M$ in (12) and (13) then there would be 2M equations in the 2M unknowns a_n and x_{2j}. It is found, values of a_n so determined give a poor fit in intervals between data points. Over-determination of the system of equations (12) and (13), by letting $N < M$, and utilizing a least-squares fit leads to a smooth approximation. Equations (12) and (13) may be rewritten in the least-squares sense as

$$X_{1j} = \sum_{n=1}^{N} a_n \cos[(3-2n)x_{2j}] + F_j \tag{14}$$

$$X_{2j} = \sum_{n=1}^{N} a_n \sin[(3-2n)x_{2j}] + H_j \tag{15}$$

where F_j and H_j are error terms. The total error E is given by

$$E = \sum_{n=1}^{M} (F_j^2 + H_j^2) \tag{16}$$

Minimization of E requires that

$$\frac{\partial E}{\partial a_n} = 0 \qquad j = 1,N \tag{17}$$

$$\frac{\partial E}{\partial x_{2j}} = 0 \qquad j = 1,N \tag{18}$$

Then

$$\frac{\partial E}{\partial a_k} = \sum_{j=1}^{M} X_{1j} \mathrm{Cos}[(3-2k)x_{2j}] + X_{2j} \mathrm{Sin}[(3-2k)x_{2j}]$$

$$- \sum_{j=1}^{M} \sum_{n=1}^{N} a_n \cos[2(k-n)x_{2j}] = 0 \qquad k = 1,N \tag{19}$$

The Lewis forms provide an estimate of a_i, $i = 1,3$; see [1]. Assuming that $a_i = 0$, $i = 4,N$. A corresponding set of values of x_{2j} which minimizes E may be found by utilizing these values. This was accomplished through a search method. By utilizing this set of values for x_{2j}, a new set of values for a_n may be computed from equation (19). An iterative procedure is followed in which the (n+1) approximation is found from the (n) approximation by letting $a_{n+1} = 0$, $a_{n+2} = 0$, ..., and a_n for $n \leq n+1$, be the coefficient of the best n term approximation. Fig. 1 is an example of the approximation of the cross-section of a circular cylindrical cavity by the mapping function retaining one term in (10). Fig 2 shows the mapping of the interior of the cross-section of a non-circular cylindrical cavity composed of vertical line segments which are tangents to semi-circular arcs. Five terms were retained for the approximation of this geometry. The mapping of the interior of the unit circle onto the interior of the cross-section of a hexagonal cylindrical cavity is shown in Fig. 3. Twenty terms were retained in the series. Application of the technique to geometries of greater complexity than the foregoing will generally require that more terms are retained in the mapping function (10).

3. FINITE MATHEMATICAL DOMAIN

To map the infinite domain exterior to unit circle in the z-plane, defined by $x_1 \geq 1$, $-\pi \leq x_2 \leq +\pi$, onto the finite mathematical domain $0 \leq y \leq 1$, $-\pi \leq x_2 \leq +\pi$, as shown in Fig. 4(a) and Fig. 4 (b), let

$$x_1 = 1/y \tag{20}$$

Equations (3) and (4) can then be expressed in terms of y and x_2 by using (11) and (20) to give :

$$c_1^2 y^2 \frac{\partial}{\partial y}[(y^2/g_{11})(\frac{u}{y} - \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x_2})]$$

$$-c_2^2 y \frac{\partial}{\partial x_2}[(y/g_{11})(y^2 \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x_2})] + \frac{\partial^2 u}{\partial t^2} = 0 \tag{21}$$

$$c_1^2 \frac{\partial}{\partial x_2}[(y^2/g_{11})(\frac{u}{y} - \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x_2})]$$

$$+c_2^2 y \frac{\partial}{\partial y}[(y/g_{11})(y^2 \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x_2})] - \frac{\partial^2 v}{\partial t^2} = 0 \tag{22}$$

Let us now introduce the scalar wave equation

$$\nabla^2\phi + \beta_1^2\phi = 0 \tag{23}$$

where $\beta_1 = \omega/c_1$. A solution to equations (21) and (22), in terms of two scalar wave function ϕ and ψ, which is valid for orthogonal cylindrical curvilinear coordinates, is given by

$$u = y\frac{\partial\psi}{\partial x_2} - y^2 \frac{\partial\phi}{\partial y} \tag{24}$$

$$v = y\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial x_2} \tag{25}$$

where ϕ satisfies the scalar wave equation (23) and ψ satisfy (23) with β_1 replaced by $\beta_2 = \omega/c_2$, see [2,3].

The problem of solving equations (1) and (2) in the infinite physical domain, has been transformed into the problem of solving two scalar wave equations in the finite mathematical domain $0 \leq y \leq 1$, $-\pi \leq x_2 \leq +\pi$.

4. FINITE-DIFFERENCE ANALYSIS

The finite - difference approximate solution for the infinite elastic medium must satisfy a radiation condition at infinity. For the wave equation this requirement is the Sommerfeld radiation condition. See [3]

To assure an outgoing wave, let

$$\phi = f\, e^{(\alpha_1/y)} \tag{26}$$

$$\psi = h\, e^{(\alpha_2/y)} \tag{27}$$

where $\alpha_1 = ia_1\,\beta_1$, $\alpha_2 = ia_1\,\beta_2$, $f(y,x_2)$ and $h(y,x_2) = 0$ for $y = 0$ and a_1 is the first term in (10). See [4]. Equations (26) and (27) may be used to eliminate ϕ and ψ from (23-25). Then equation (23) becomes

$$\frac{\partial^2 f}{\partial y^2} + (y^{-1} - 2\alpha_1 y^{-2})\frac{\partial f}{\partial y} + (\alpha_1 y^{-3} + \alpha_1^2\, y^{-4} + g_{11} y^{-4}\beta_1^2)f$$

$$+ y^{-2}\, \frac{\partial^2 f}{\partial x_2^2} = 0 \tag{28}$$

The function $h(y,x_2)$ associated with ψ also satisfy equation (28) with identical coefficients except that α_1 and β_1 are replaced with α_2 and β_2. Similarly the dependent variable ϕ and ψ in (24) and (25) may also be replaced with f and h to give

$$u = (ye^{\alpha_2/y})\frac{\partial h}{\partial x_2} + (\alpha_1 e^{\alpha_1/y})f - (y^2 e^{\alpha_1/y})\frac{\partial f}{\partial y} \qquad (29)$$

$$v = (e^{\alpha_2/y})\frac{\partial h}{\partial y} - ((\alpha_2/y)e^{\alpha_2/y})h + (e^{\alpha_1/y})\frac{\partial f}{\partial x_2} \qquad (30)$$

For the finite-difference approximate solution, the rectangular finite domain $0 \leq y < 1$, $-\pi \leq x_2 \leq \pi$, of Fig. 4 (b) may be subdivided into a grid with N subdivisions in the y direction and M subdividisions in the x_2 direction. The spacings along the y and x_2 directions are then given respectively by

$$\delta_1 = 1/N \qquad (31)$$

$$\delta_2 = 2\pi/M \qquad (32)$$

The derivatives of equation (28) may be replaced by the central-difference expansions with errors of order δ_1^2 and δ_2^2. At a general nodal point, $y = y_1$ and $x_2 = x_{2k}$, the finite-difference form of (28) is

$$Q_1 \cdot f_{i+1,k} + Q_2 \cdot f_{i,k} + Q_3 \cdot f_{i-1,k} + Q_4 \cdot f_{i,k+1}$$

$$+ Q_5 \cdot f_{i,k-1} = 0 \qquad (33)$$

where

$$Q_1 = [(1/\delta_1^2) + (1/y_i - 2\alpha_1/y_i^2)/(2\delta_1)]$$

$$Q_2 = [(\alpha_1/y_i^3) + (\alpha_1^2 + g_{11}\beta_1^2)/y_i^4 + 2(1/\delta_1^2 + 1/y_i^2\delta_2^2)]$$

$$Q_3 = [(1/\delta_1^2) - (1/y_i - 2\alpha_1/y_i^2)/(2\delta_1)]$$

$$Q_4 = [1/(y_i^2\delta_2^2)]$$

$$Q_5 = [1/(y_i^2\delta_2^2]$$

Finite difference forms of (29) and (30) may be readily derived by replacing first derivatives with a central-difference approximation.

5. BOUNDARY CONDITIONS

For specified time-harmonic tractions applied to the surface of the embedded cylindrical cavity, the stress boundary conditions on the internal boundary of the infinite physical domain will be given over disjoint set of points in terms of the physical components, T_i, of the stress vector, and, N_i, of the exterior unit normal vector associated with the rectangular Cartesian coordinate X_i. In terms of the coordinates of the mathematical finite region, the components of the unit normal vector are

$n_1 = -y^2 (\partial X_j/\partial y)N_j$ and $n_2 = (\partial X_j/\partial x_2)N_j$; while the components of the stress vector are $t_1 = -y^2 (\partial X_j/\partial y)T_j$ and $t_2 = (\partial X_j/\partial x_2)N_j$.

In the x_i corodinate system

$$t_i = (\lambda g_{ij} e_{kk} + 2\mu e_{ij}) n^j \qquad (34)$$

where $e_{ij} = (u_{j;i} + u_{i;j})/2$, i,j = 1,2 and a semicolon represents covariant partial differentiation. Since t_i and n_i are given at points where the stress boundary condition is satisfied and e_{ij} may be expressed as a function of u_i, it follows that (34) provides two equations relating f and h on stress boundaries.

6. NUMERICAL STUDEIS

The numerical technique was applied to the problem of a homogeneous, isotropic unlayered elastic medium of infinite extent containing a single infinite cylindrical cavity of radius a(m), which is subjected to time-harmonic tractions applied to the entire surface of the cavity, so that an axially symmetric elastic dilatational pulse is propagated outwards into the elastic medium. The pulsatance is at ω radians per second.

For simplicity we assume here that the uniform normal loading resulting in an axially symmetric dilatational pulse may be considered as that due to an attached mass M per unit length, and a reduced forcing function F_o acting on the mass. For points in the elastic medium, it is easily shown that the exact displacements are

$$u = (F_o/\Delta)\, H_1^{(1)}(\beta_1 x_1) \qquad (34)$$

$$v = 0 \qquad (35)$$

where $\Delta = (\lambda+2G)(\beta_1/2)[H_0^{(1)}(\beta_1) - H_2^{(1)}(\beta_1)] + (\lambda-M\omega^2)H_1^{(1)}(\beta_1)$,

x_1 is the radial distance from the centre of the cavity, and $H_n^{(1)}(z)$, n = 0, 1, 2 are Hankel functions of the first kind.

Because the problem is axi-symmetric, a one-dimensional solution along any radius is sufficient. Nevertheless, a two-dimensional solution was implemented to illustrate the method of solution in two-dimensions. For the geometry under consideration, the coefficients occuring in the expansion of the mapping function (10) are $a_1 = 1.0$, $a_n = 0.0$ for $n > 1$. All calculations were performed using the following material constants: $\rho = 7850$ Kg/m^3, $\nu = 0.3$, G = 79.6 GN/m^2 and E = 207 GN/m^2, which correspond to a mild steel medium. The other parameters are $F_o = 1.0$, $\omega = 500.0$,

$$\delta_1 = 0.01,\ \delta_2 = \pi/10,\ M = 1,\ -\pi \le x_2 \le \pi,\ 0 \le y \le 1.0 \text{ and } a = 1.0.$$

The magnitude of the displacements and the phase angles along a radius, for the exact solutions and the numerical approximations are illustrated in Fig. 5 and Fig. 6 respectively. The agreement between the exact solutions and those obtained by the numerical technique outlined here-in is seen to be excellent. The maximum error is less than 0.01 per cent.

7. CONCLUSIONS

A numerical procedure for the evaluation of the displacement field generated in an infinite unlayered elastic medium, by the application of time-harmonic tractions to the surface of a single embedded infinite cylindrical cavity of arbitrary cross-section has been given. Numerical solutions for a circular cross-section are given as an example of this approach. The agreement between the exact solutions and those obtained by the numerical technique is seen to be excellent.

In addition to problems in steady elastic vibrations, this procedure appears feasible for the numerical computation of surface deformation associated with volcanism, the investigation of shallow-cylindrical shells containing irregular holes or boundaries, and infinite systems occuring in soil dynamics, seismology and acoustic generators.

REFERENCES

1. VON KERCYEK, C., and TUCK, E.O., "The Representation of Ship Hulls by Conformal Mapping Functions",

journal of Ship Research, Dec. 1969, pp. 284—298.
2. CHADWICK, P., and TROWBRIDGE, E.A., "Elastic Wave Fields Generated by Scalar Wave Functions", Proc. Camb. Phil. Soc., Vol. 63, 1967, pp. 1177—1187.
3. ERINGEN, A.C., and SUHUBI, E.S., Elastodynamics, Vol. II, Academic Press, New York, 1975.
4. BERGER, B.S., and ALABI, B., "Steady-State Solution of Navier's Equation in Cylindrical Curvilinear Coordinates", ASME Journal of Applied Mechanics, Vol. 47, Sept. 1980, pp. 682—683.
5. BERGER, B.S., and ALABI, B. "Solution of Navier's Equation in Cylindrical Curvilinear Coordinates", ASME Journal of Applied Mechanics, Vol. 45, Dec. 1978, pp. 812—816.
6. CHREE, C., "The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Coordinates, Their Solution and Applications", Camb. Phil. Soc. Trans, Vol. 14,1889, pp. 250—321.
7. DIETERICH, J.H., and DECKER, R.W., "Finite Element Modelling of Surface Deformation Associated With Volcanism". Journal of Geophysical Research, Vol. 80, No. 29, Oct. 1975, pp. 4094—4102.
8. IVES, D.C., "A Modern Look at Conformal Mapping Including Multiply Connected Regions", Jour. of AIAA, Vol. 14, 1976, pp. 1006—1011.
9. LYSMER, T., and WASS, G., "Shear Waves in Plane Infinite Structures", Jour Eng. Mechs. Div., ASCE, Vol. 98, Feb. 1972, pp. 85—105.
10. SYNGE, J.L., and SCHILD, A., Tensor Calculus, University of Toronto Press, 1962.
11. ALABI, B., "A Mapping Finite-Difference Model for Infinite Elastic Media", (To be published in the Journal of Applied Mathematical Modelling).

LIST OF SYMBOLS

a_n Coefficients of the mapping function
c_1 Propagation velocity of dilatation waves.
c_2 Propagation velocity of distortional waves.
f A function of y and x_2.
g_{ij} Components of the metric tensor.
G Lamé's elastic constant.
h A function of y and x_2.
i Imaginary unit $\sqrt{-1}$.
t Physical time.
u_i Displacement components in the x_i direction.
U_i Displacement component in the X_i direction.
x_i Orthogonal curvilinear coordinates.
X_i Rectangular Cartesian coordinates.
y $= 1/x_1$.
ϕ Scalar wave function.
ω Pulsalance in radians per second.
λ Lamé's elastic constant.
ψ Scalar wave function.

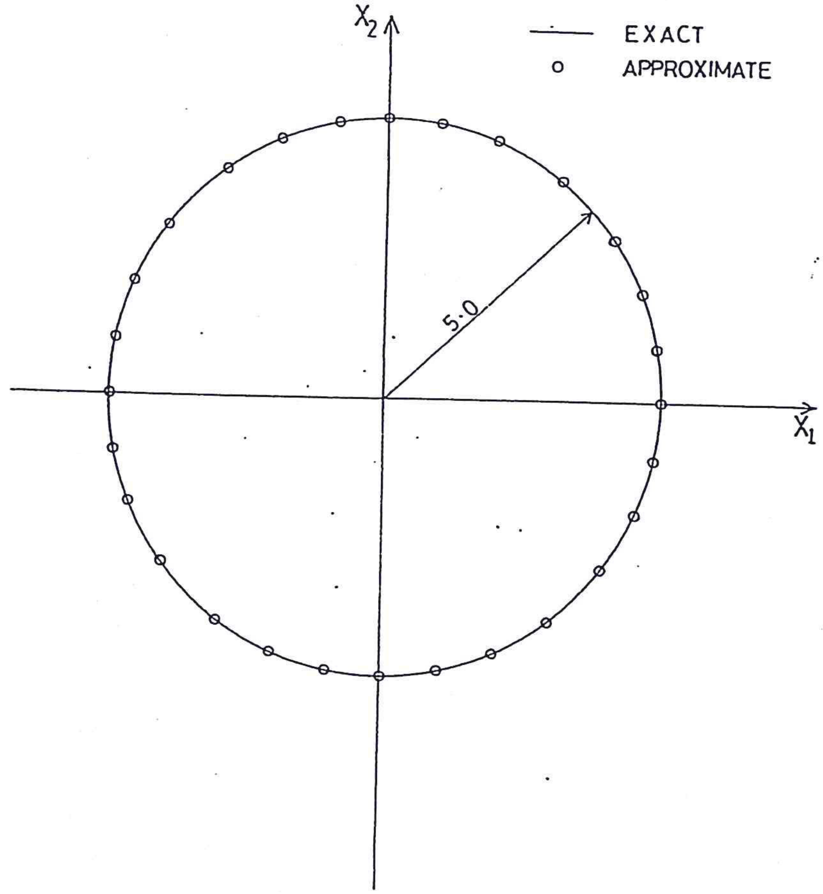

Fig. 1: Mapping of the interior of the unit circle onto the cross-section of a circular cylindrical cavity

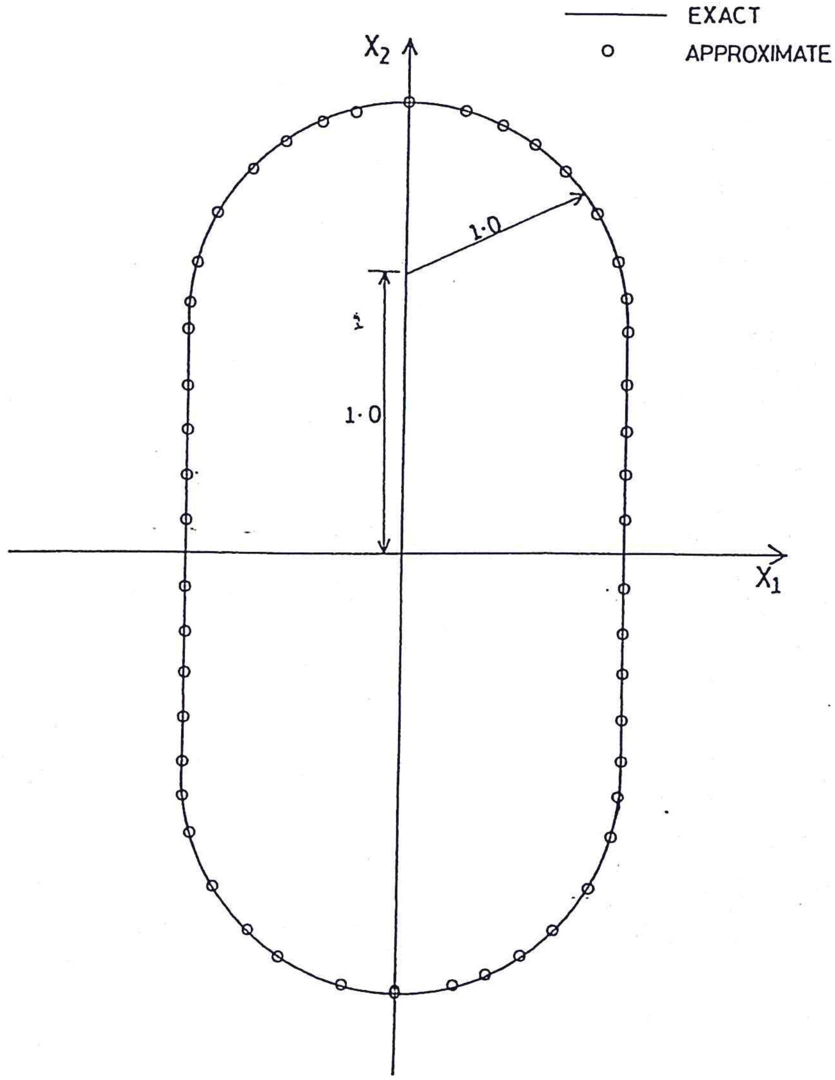

Fig. 2: Mapping of the interior of the unit circle onto the cross-section of a non-circular cylindrical cavity

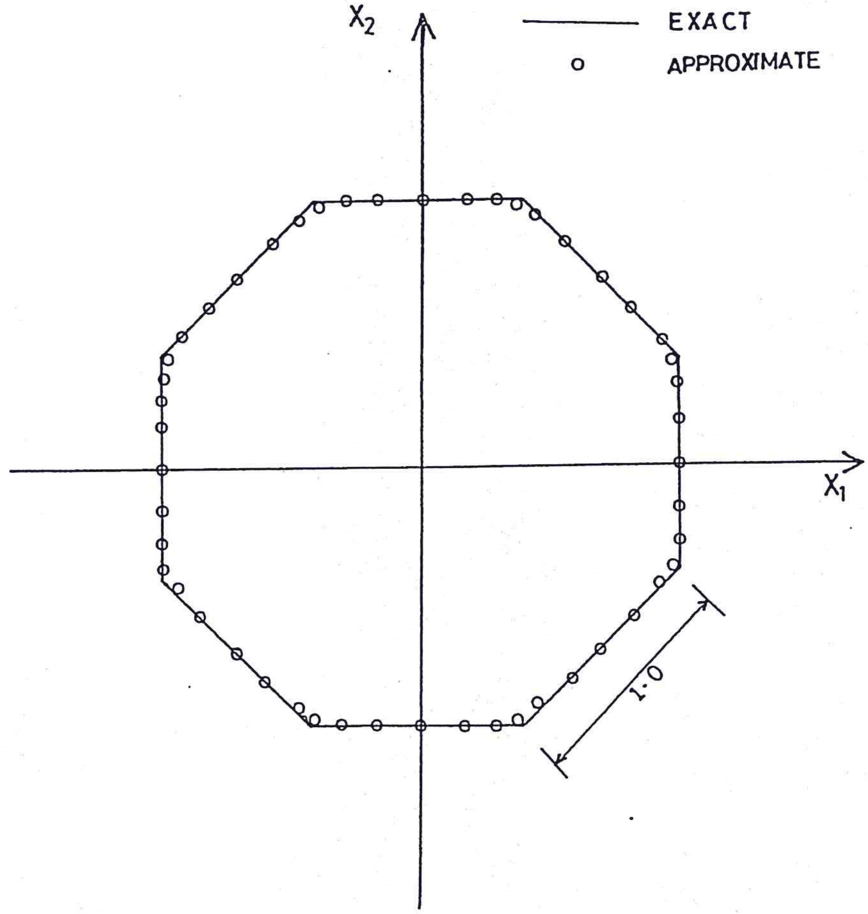

Fig. 3: Mapping of the interior of the unit circle onto the cross-section of a hexagonal cylindrical cavity

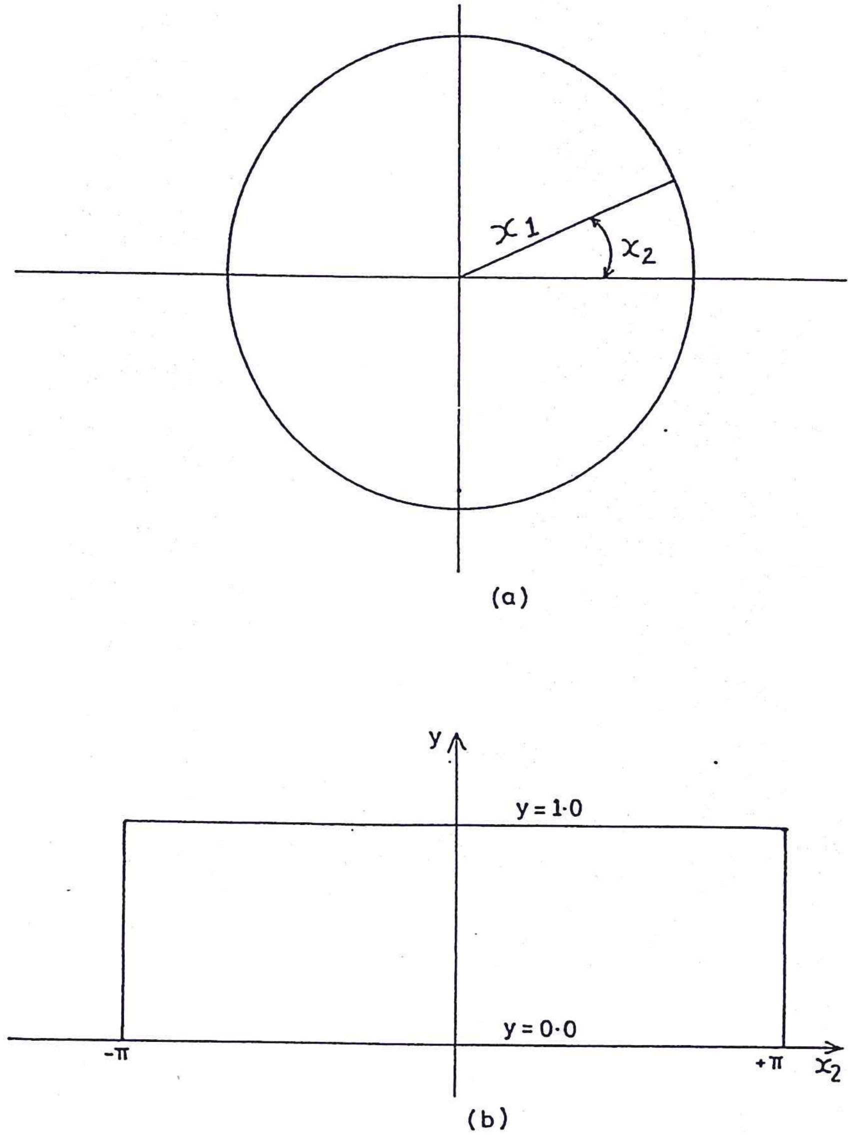

Fig. 4 : (a) The z - plane with co-ordinates x_1 and x_2

(b) The finite mathematical domain

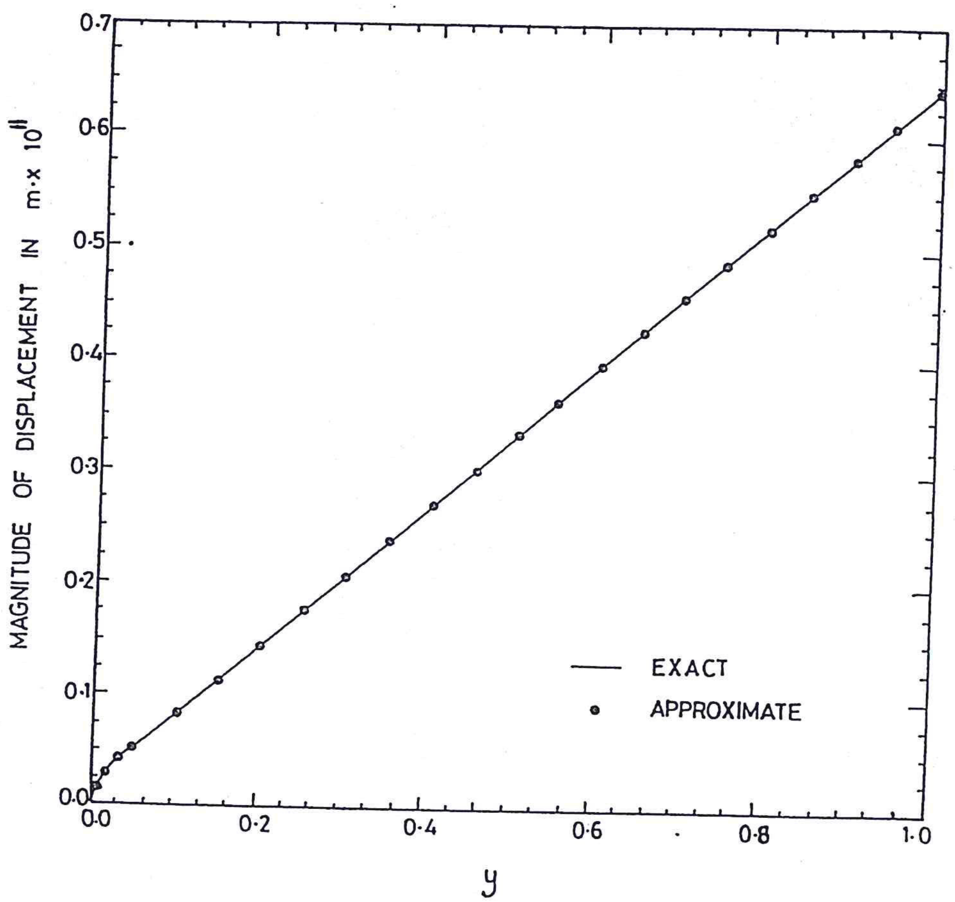

Fig. 5 : Comparison of the magnitude of the displacements obtained from exact solution and from numerical method

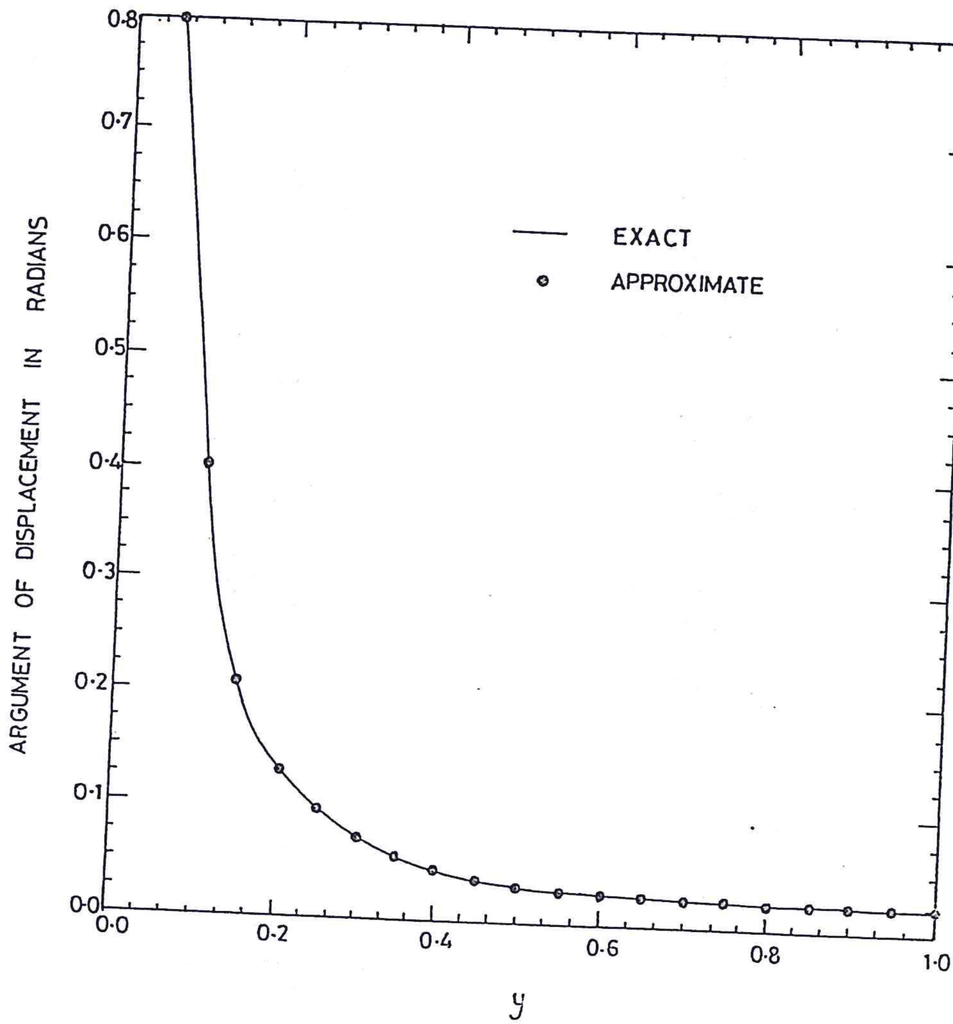

Fig. 6 : Comparison of the argument of the displacements obtained from exact solution and from numerical method

SHORTER COMMUNICATIONS