Analytical Solutions Applied To Laminar And Turbulent Free Convection Boundary Layers On A Vertical Heated Plate-Comparison With Experimental Measurements

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Analytical solutions have been applied to laminar and turbulent free convection from isothermal vertical heated plate. The momentum and energy equations in integral form are solved simultaneously to obtain the boundary layer thickness and heat transfer coefficient. For the laminar boundary layer, polynomial profiles are assumed for velocity and temperature distributions to satisfy the hydrodynamic and thermal boundary conditions. For the turbulent boundary layer, one-seventh power laws are assumed for velocity and temperature distributions. The results of the analytical techniques are compared with the established experimental measurements in the literature.

Keywords: Analytical solution; Free convection; Laminar boundary layer; Turbulent boundary layer

NOTATION

a, b, c, d	Constants in velocity distribution (Eq. 5)	u', v'	Fluctuation velocity components
a', b', c', d'	Constants in temperature distribution (Eq.8)	$\overline{u}$	Mean streamwise velocity
c_1, m	Constants (Eq. 12)	$\overline{u'v'}$	Reynolds shear stress
c_2, n	Constants (Eq. 13)	u_x	Reference velocity (Eq. 7)
c_p	Specific heat at constant pressure	u_1	Reference velocity (Eq. 34)
C_f	Skin friction factor	U	Dimensionless mean streamwise velocity
g	Gravitational acceleration	u^*	Reference velocity $[g\beta\Delta Tx]^{1/2}$
h	Heat transfer coefficient	v	Velocity component in y direction
Gr	Grashof number	x	Coordinate in flow direction
k	Thermal conductivity	y	Coordinate normal to the flow
L	Plate length	α	Thermal diffusivity
Nu	Nusselt number	β	Volumetric expansion coefficient
Pr	Prandtl number	ρ	Density
q_w	Heat flux from (or to) the wall	τ_w	Shear stress at the wall
Ra=Gr.Pr	Rayleigh number	υ	Kinematic viscosity
Re	Reynolds number	η	Dimensionless vertical distance
T	Local flow temperature	θ	Dimensionless temperature difference
T_∞	Ambient temperature	μ	Dynamic viscosity
T_w	Wall temperature	δ	Hydrodynamic boundary layer thickness
u	Velocity component in x direction	δ_T	Thermal boundary layer thickness

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1. Introduction

Free convection systems have been the focus of interest since the middle of the last century. The problems of natural convection in both laminar and turbulent flows have been investigated extensively over the last five decades. Most of these studies are numerical in nature, and the experimental ones have reported mainly the heat transfer and the mean turbulent quantities. None of these studies have reported detailed measurements of the time-mean and fluctuating flow and thermal fields in turbulent mixed-convection flow over a vertical flat plate, but, some valuable experimental measurements are released in the literature recently. Abu-Mulaweh et al [1] performed measurements to study the laminar mixed convection adjacent to a vertical plate in the case of uniform wall heat flux. Qiu et al [2] evaluated the local wall temperature, heat flux, and convective heat transfer coefficient from the near wall temperature profile. Patel et al [3] studied the transition from turbulent natural to turbulent forced convection adjacent to an isothermal vertical plate. Abu-Mulaweh et al [4] studied the turbulent natural convection flow over a vertical backward facing step experimentally. Abu-Mulaweh et al [5] studied the effect of free stream velocity on turbulent natural convection flow along vertical plate experimentally. They reported detailed measurements of time-mean velocity and temperature distributions, intensities of the distributions of velocity and temperature fluctuations, Reynolds shear stress, and local Nusselt number distributions in turbulent mixed convection flow over a vertical flat plate. They also studied the free-stream velocity on turbulent natural convection flow along a vertical flat plate. In this paper consideration has been given to a boundary layer convection problem where the solution procedure is not unduly complicated when simplifying assumptions are made. The analytical solution has been applied to approximately solve the integral equations. The hydrodynamic and thermal boundary layers are analyzed for fluids with Prandtl number near unity so that the thickness of two layers can be assumed to be equal. The velocity and temperature distributions are developed making reasonable assumptions for the profiles. The coupled momentum and energy equations are solved simultaneously to give the boundary layer thickness and the heat transfer coefficient. Local velocities, shear stresses, temperatures, and heat transfer coefficients are numerically evaluated from the analytical solution. They are then compared with those measured experimentally [5].

The governing equations describing the hydrodynamic and thermal characteristics of free convection heat transfer from an isothermal, vertical heated plate are the momentum and energy equations which can be obtained from two different approaches, namely, differential approach and integral approach. In differential approach the equations are derived for an infinitesimal fluid element, while in integral approach the equations are derived for a finite control volume [6]. The momentum and energy equations are coupled, meaning that both equations are expressed in terms of velocity and temperature fields and therefore have to be solved simultaneously.

The momentum and energy equations for the forced convection systems are independent, that is, the momentum equation is expressed in terms of velocity field only, but the energy equation is expressed in terms of both velocity and temperature fields. Therefore the momentum equation can be solved independently for the velocity distribution. The velocity distribution can then be inserted in the energy equation, and the energy equation can be solved for the temperature distribution.

2. Laminar Free Convection From a Vertical Plate

The momentum and energy equations for the natural convection system on a vertical heated flat plate may be derived based on differential or integral methods. The governing equations in differential form are as follows [7]

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+\upsilon\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \quad (2)$$

The above equations have been derived based on the following assumptions [7]:

1. Incompressible and steady flow,
2. Constant thermal properties,
3. Negligible heat conduction in the direction of gravity,
4. Negligible viscous work done on the fluid element, and
5. Constant free stream temperature.

The governing equations in integral form are as follows [7]

$$\frac{d}{dx}\int_0^\delta \rho u^2 dy = -\mu\frac{\partial u}{\partial y}]_{y=0}+\int_0^\delta \rho g\beta(T-T_\infty)dy \quad (3)$$

$$\frac{d}{dx}\int_0^\delta u(T-T_\infty)dy=-\alpha\frac{dT}{dy}]_{y=0} \quad (4)$$

The assumptions listed for the differential equations have been exactly applied for derivation of integral equations.

2.1 Boundary conditions

The hydrodynamic and thermal boundary conditions for the natural convection system on the vertical heated plate are listed in Table 1.

Table 1: *Hydrodynamic and thermal boundary conditions in laminar natural convection from a vertical isothermal plate*

y	Hydrodynamic	Thermal
0	$u=0$ $\frac{\partial^2 u}{\partial y^2}=-\frac{g\beta}{\upsilon}(T_w-T_\infty)$	$T=T_w$ $\frac{\partial^2 T}{\partial y^2}=0$
δ	$u=0$ $\frac{\partial u}{\partial y}=0$	$T=T_\infty$ $\frac{\partial T}{\partial y}=0$

2.2 Velocity distribution

A polynomial profile of the third order with respect to y is assumed to represent the velocity distribution,

$$u=a+by+cy^2+dy^3 \quad (5)$$

Applying the hydrodynamic boundary conditions into the above profile, the constants of the profile are determined and then inserted into Eq. 5. The dimensionless velocity distribution becomes:

$$\frac{u}{u_x}=\frac{y}{\delta}(1-\frac{y}{\delta})^2 \quad (6)$$

where,

$$u_x=\frac{\beta\delta^2 g(T_w-T_\infty)}{4\upsilon} \quad (7)$$

2.3 Temperature distribution

A polynomial profile of the third order with respect to y is assumed to represent the temperature distribution,

$$T=a'+b'y+c'y^2+d'y^3 \quad (8)$$

Applying the thermal boundary conditions into the above profile, the constants of the profile are determined and then inserted into Eq. 8. The dimensionless temperature distribution becomes:

$$\frac{T-T_w}{T_\infty-T_w}=\frac{3}{2}(\frac{y}{\delta})-\frac{1}{2}(\frac{y}{\delta})^3 \quad (9)$$

2.4 Solution method

The velocity and temperature distributions as expressed in Eqs. 8 and 9 are inserted into the integral momentum and energy equations (Eqs. 3 and 4), resulting in:

$$\frac{1}{105}\frac{d}{dx}(u_x^2\delta)=\frac{3}{8}g\beta(T_w-T_\infty)\delta-\upsilon\frac{u_x}{\delta} \quad (10)$$

$$\frac{4}{105}(T_w-T_\infty)\frac{d}{dx}(u_x\delta)=\frac{3\alpha}{2\delta}(T_w-T_\infty) \quad (11)$$

An analytical solution exists for the above equations, when answers are assumed to be of the following form:

$$u_x=c_1x^m \quad (12)$$

$$\delta = c_2 x^n \qquad (13)$$

where c_1, c_2, m, and n are constants. Substitution of Eqs. 12 and 13 into Eqs. 10 and 11 yields:

$$\frac{1}{105} c_1^2 c_2 (2m+n) x^{2m+n-1} = \frac{3}{8} g\beta (T_w - T_\infty) c_2 x^n - \upsilon \frac{c_1}{c_2} x^{m-n} \qquad (14)$$

$$\frac{4}{105} c_1 c_2 (m+n) x^{m+n-1} = \frac{3\alpha}{2c_2} x^{-n} \qquad (15)$$

Analytical solution exists, only if both sides of these equations are independent of x, thus the exponents m and n must be related by:

$$2m + n - 1 = n = m - n \qquad (16)$$

$$m + n - 1 = -n \qquad (17)$$

or,

$$m = \frac{1}{2} \quad and \quad n = \frac{1}{4}$$

Simultaneous solution of Eqs. 14 and 15 for the coefficients c_1 and c_2 provides the following results:

$$c_1 = 4.44\, \upsilon \left(\frac{5}{8} + \frac{\upsilon}{\alpha}\right)^{-1/2} \times \left[\frac{g\beta (T_w - T_\infty)}{\upsilon^2}\right]^{1/2} \qquad (18)$$

$$c_2 = 3.44 \left(\frac{5}{8} + \frac{\upsilon}{\alpha}\right)^{1/4} \times \left[\frac{g\beta (T_w - T_\infty)}{\upsilon^2}\right]^{-1/4} \left(\frac{\upsilon}{\alpha}\right)^{-1/2} \qquad (19)$$

Consequently,

$$\frac{\delta}{x} = c_2 x^{n-1} = 3.44 (0.625 + \Pr)^{1/4} Gr_x^{-1/4} \Pr^{-1/2} \qquad (20)$$

$$u_x x = c_1 x^{m+1} = 4.44\, \upsilon (0.625 + \Pr)^{-1/2} Gr_x^{1/2} \qquad (21)$$

where Gr_x is the local Grashof number and Pr is the Prandtl number,

$$Gr_x = \frac{g\beta (T_w - T_\infty) x^3}{\upsilon^2} \qquad (22)$$

$$\Pr = \frac{\upsilon}{\alpha} = \frac{\mu c_p}{k} \qquad (23)$$

δ and u_x are now inserted into Eqs. 6 and 9 to obtain the velocity and temperature profiles.

2.5 Heat transfer coefficients in laminar boundary layer

The local Nusselt number is defined as:

$$Nu_x = \frac{q_w x}{k (T_w - T_\infty)} \qquad (24)$$

where,

$$q_w = -k \left[\frac{\partial T}{\partial y}\right]_{y=0} = \frac{3}{2} k \frac{T_w - T_\infty}{\delta} \qquad (25)$$

Then,

$$Nu_x = \frac{3}{2} \frac{x}{\delta} \qquad (26)$$

Using Eq. 20 gives,

$$Nu_x = 0.436 (0.625 + \Pr)^{-1/4} Gr_x^{1/4} \Pr^{1/2} \qquad (27)$$

Since Nu_x is proportional to $x^{3/4}$, the mean Nusselt number over the entire plate length is:

$$\overline{Nu} = \frac{1}{L} \int_0^L Nu_x dx = \frac{4}{3} Nu_{x=L} \qquad (28)$$

or,

$$\overline{Nu} = 0.581 (0.625 + \Pr)^{-1/4} Gr_L^{1/4} \Pr^{1/2} \qquad (29)$$

Eq. 29 which was developed for fluids with Prandtl numbers in the order of unity, can be used for air with Pr=0.71, it then simplifies to:

$$\overline{Nu} = 0.456\, Gr_L^{1/4} \qquad (30)$$

The analytically calculated mean Nusselt numbers, agree well with the exact solution calculated numerically by Schmidt and Beckman [8] as:

$$\overline{Nu} = 0.48\, Gr_L^{1/4} \qquad (31)$$

The difference between two approaches is only 5%.

3. Turbulent Free Convection from a Vertical Plate

It was demonstrated that the apparently gross assumptions of the integral techniques of analysis, yield reasonably accurate predictions for laminar flows. Attention will now be turned to the more common turbulent regime. It is, however, necessary to consider what governs the transition from laminar to turbulent flow in free convection. Experimental values of mean Nusselt numbers against the Raleigh numbers as measured by Eckert and Jackson [9] show that for $Ra<10^9$, the results are correlated by $\overline{Nu}=0.555Ra_L^{1/4}$ while for $Ra>10^9$, the results are correlated by $\overline{Nu}=0.021Ra_L^{2/5}$. It can be seen, there is a change of slope around $Ra_L=10^9$, this corresponds to transition from laminar to turbulent flow. Therefore, for $Ra_L>10^9$ heat transfer from a vertical heated plate would be controlled by turbulent flow.

The same method of analysis is used as that for laminar boundary layer, with the exception that the velocity and temperature profiles are presented by the one-seventh power laws. It is still assumed that viscous dissipation is negligible and $\delta_T \approx \delta$. For the case of the isothermal plate with temperature T_w and a quiescent free stream with constant temperature T_∞, Eqs. 3 and 4 can be used for turbulent free convection, as:

$$\frac{d}{dx}\int_0^\delta u^2 dy = g\beta \int_0^\delta (T-T_\infty)dy - \frac{\tau_w}{\rho} \qquad (32)$$

$$\frac{d}{dx}\int_0^\delta u(T-T_\infty)dy = \frac{q_w}{\rho c_p} \qquad (33)$$

The one-seventh power law profiles are postulated for velocity and temperature distributions [9] as:

$$u = u_1 \eta^{1/7}(1-\eta)^4 \qquad (34)$$

and

$$\theta = 1 - \eta^{1/7} \qquad (35)$$

where,

$$\eta = \frac{y}{\delta} \quad and \quad \theta = \frac{T-T_\infty}{T_w - T_\infty}$$

The Blasius expression [10] for shear stress in turbulent forced convection pipe flows, is modified for external flows and used for turbulent free convection over a flat plate as:

$$\frac{\tau_w}{\rho} = 0.0225\, u_1^2 \left(\frac{\upsilon}{u_1\delta}\right)^{1/4} \qquad (36)$$

As Prandtl numbers of order unity are considered, the modified Reynolds analogy [11] can be used as:

$$Nu_x = \frac{1}{2}\Pr^{1/3} Re_x C_f \qquad (37)$$

which implies that:

$$\frac{q_w}{k(T_w - T_\infty)} = \frac{1}{2}\Pr^{1/3}\left(\frac{u_1 x}{\upsilon}\right)\left(\frac{\tau_w}{\frac{1}{2}\rho u_1^2}\right)$$

or

$$\frac{q_w}{\rho c_p} = \Pr^{-2/3}\frac{T_w - T_\infty}{u_1}\left(\frac{\tau_w}{\rho}\right) \qquad (38)$$

Substitution of Eqs. 34, 35, 36 and 38 into Eqs. 32 and 33 produces the following results:

$$0.0523\frac{d}{dx}(u_1^2\delta) = 0.125\, g\beta(T_w - T_\infty)\delta - 0.0225\, u_1^2\left(\frac{\upsilon}{u_1\delta}\right)^{1/4} \qquad (39)$$

$$0.0366 \frac{d}{dx}(u_1\delta) = 0.0225 \ \mathrm{Pr}^{-2/3} u_1 (\frac{\upsilon}{u_1\delta})^{1/4} \quad (40)$$

By analogy with the laminar free convection case, it is proposed to solve these coupled differential equations by similarity solutions of the form:

$$u_1 = c_1 x^m \quad and \quad \delta = c_2 x^n$$

Using these substitutions, Eqs. 39 and 40 become:

$$0.0523 \frac{d}{dx}(c_1^2 c_2 x^{2m+n}) = 0.125\, g\beta(T_w - T_\infty) \times c_2 x^n - 0.0225\, (c_1 x^m)^{7/4} (c_2 x^n)^{-1/4} \upsilon^{1/4} \quad (41)$$

$$0.0366 \frac{d}{dx}(c_1 c_2 x^{m+n}) = 0.0225 \ \mathrm{Pr}^{-2/3} (c_1 x^m)^{3/4} (c_2 x^n)^{-1/4} \upsilon^{1/4} \quad (42)$$

In order that analytical solutions exist, it is obviously necessary that:

$$2m + n - 1 = n = \frac{7}{4}m - \frac{1}{4}n$$

and,

$$m + n - 1 = \frac{3}{4}m - \frac{1}{4}n$$

hence,

$$m = 0.5 \quad and \quad n = 0.7$$

After some straightforward, but tedious, manipulation of Eqs. 41 and 42, it can be shown that:

$$c_1 = 0.0689 \ c_2^{-5} \upsilon \ \mathrm{Pr}^{-8/3} \quad (43)$$

$$c_2 = [0.00338 \frac{\upsilon^2}{g\beta(T_w - T_\infty)} \times (1 + 0.494 \ \mathrm{Pr}^{2/3}) \mathrm{Pr}^{-16/3}]^{1/10} \quad (44)$$

Using the fact that $u_1 = c_1 x^{0.5}$ and remembering that the local Grashof number, Gr_x, is defined according to Eq. 21, u_1 can be written in the following form:

$$u_1 = 1.185 \frac{\upsilon}{x} Gr_x^{1/2} \times (1 + 0.494 \ \mathrm{Pr}^{2/3})^{-1/2} \quad (45)$$

Similarly, using the fact that $\delta = c_2 x^{0.7}$, the boundary layer thickness can be expressed as:

$$\frac{\delta}{x} = 0.565 \ Gr_x^{-1/10} \ \mathrm{Pr}^{-8/15} \times (1 + 0.494 \ \mathrm{Pr}^{2/3})^{1/10} \quad (46)$$

Eq. 37 can now be solved for the Nusselt number,

$$\mathrm{Re}_x = \frac{u_1 x}{\upsilon} = 1.185 \ Gr_x^{1/2} \times (1 + 0.494 \ \mathrm{Pr}^{2/3})^{-1/2} \quad (47)$$

and

$$\frac{1}{2} C_f = \frac{\tau_w}{\rho u_1^2} = 0.0225 \ (\frac{\upsilon}{u_1 x})^{1/4} (\frac{x}{\delta})^{1/4} \quad (48)$$

Hence, using Eqs. 45, 46, 47 and 48 it is found that :

$$Nu_x = 0.0295 \ Gr_x^{2/5} \ \mathrm{Pr}^{7/15} \times (1 + 0.494 \ \mathrm{Pr}^{2/3})^{-2/5} \quad (49)$$

The mean Nusselt number, the overall Grashof number and the mean heat transfer coefficient are defined as:

$$\overline{Nu} = \frac{\bar{h}L}{k}$$

$$Gr_L = \frac{g\beta L^3 (T_w - T_\infty)}{\upsilon^2}$$

$$\bar{h} = \frac{1}{L(T_w - T_\infty)} \int_0^L q_w \, dx$$

It can be seen from Eq. 49 that $h_x \propto x^{1/5}$. Hence, it follows that $\bar{h} = \frac{5}{6} h_L$, or

$$\overline{Nu} = 0.0246 \; Gr_L^{2/5} \Pr^{7/15} \times (1 + 0.494 \Pr^{2/3})^{-2/5} \qquad (50)$$

For air, with Pr=0.71, Eq. 50 simplifies to:

$$\overline{Nu} = 0.0183 \; Gr_L^{2/5} \qquad (51)$$

which almost agrees with the experimental results correlated by Eckert and Jackson [9] as:

$$\overline{Nu} = 0.021 \; (Gr_L \Pr)^{2/5}$$

Not surprisingly, the agreement is less satisfactory for Prandtl numbers very different from unity, and for Pr>5 the error soon exceeds 10 percent, although the analysis is quite satisfactory for gases.

3.1 Comparison of velocity and temperature distributions with experimental measurements

Abu-Mulaweh et al [5] have measured local temperatures and velocities along a vertical flat plate exposed to turbulent air flow in free convection. In this experimental study, measurements of the flow and thermal fields were carried out at one streamwise location, x=2.7 m, for a temperature difference between the heated wall and the free air stream $\Delta T = T_w - T_\infty = 30^o C$, and for a range of free stream velocities including $u_\infty = 0$. The results for zero free stream velocity are listed in Table 2, where U is the dimensionless mean streamwise velocity defined as:

$$U = \frac{u}{(g\beta\Delta T x)^{1/2}}$$

In order to make a reasonable comparison, the local velocities and temperatures are evaluated using Eqs. 34 and 35. It is then necessary to evaluate u_1 and δ at this streamwise location using Eqs. 45 and 46. For air with a corresponding Grashof number $Gr_x = 6.45 \times 10^{10}$ and Pr=0.71, Eqs. 45 and 46 are reduced to: u_1 = 1.603 m/s and $\delta / x = 0.05817$ or $\delta = 0.157\,m$. Having u_1 and δ, the velocity and temperature profiles are evaluated using Eqs. 34 and 35. The results are listed in Table 1 for the same transverse locations as used by Abu-Mulaweh et al [5]. The analytical and experimental dimensionless mean streamwise velocities are plotted against η in Fig. 1, to give a view of how the power law velocity profile agrees with the experimental data. The same comparison is made in Fig. 2 for the analytical and experimental dimensionless temperatures.

Figure 1: *Analytical and experimental velocity distributions in turbulent natural convection from a vertical heated plate*

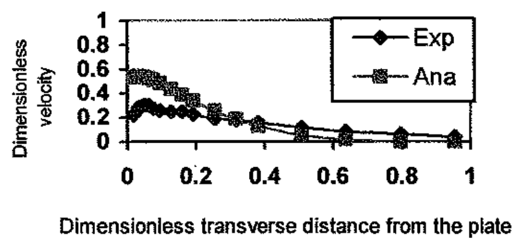
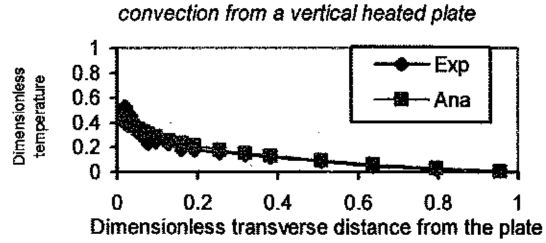


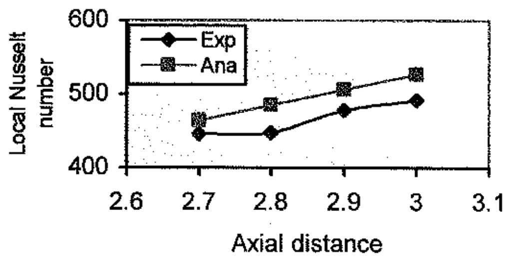
Figure 2: *Analytical and experimental temperature distributions in turbulent natural convection from a vertical heated plate*



3.2 Comparison of Nusselt numbers and shear stresses with experimental measurements

Abu-Mulaweh et al [5] measured turbulent natural convection Nusselt numbers along the vertical plate at 4 different axial locations, 2.7, 2.8, 2.9 and 3 meters from the bottom of heated plate. The local Nusselt numbers based on the analytical model presented in this paper are calculated using Eq. 49. These two sets of Nusselt numbers are comared schematically in Fig. 3.

Figure 3: *Analytical and experimental Nusselt numbers in turbulent natural convection from a vertical heated plate*



Abu-Mulaweh et al [5] have also reported detailed measurements of fluctuating velocity and Reynolds shear stress distributions in turbulent natural convection flow over a vertical flat plate. Based on these experimental data, the dimensionless shear stress in turbulent natural convection flow over a vertical flat plate at one streamwise location, x=2.7 m, for a temperature difference of 30°C between the heated wall and the free stream air, with a corresponding local Grashof number $Gr = 6.45 \times 10^{10}$, is worked out at different transverse locations from the wall, based on the following relationship:

$$\tau_{Exp} = \frac{1}{u^{*2}}(\nu \frac{\partial \overline{u}}{\partial y} - \overline{u'v'} \qquad (52)$$

where, for the above conditions, $u^* = 1.59 m/s$ and $\upsilon = 1.697 \times 10^{-5} m^2/s.$

The values of $\partial \overline{u}/\partial y$ at each transverse location is worked out based on linear approximation between that location and the previous location, namely,

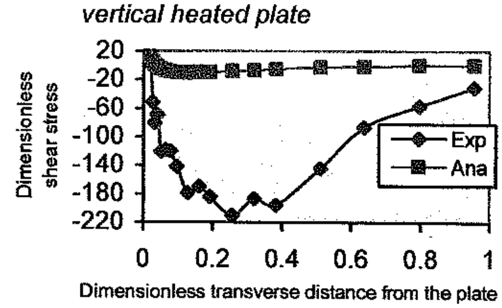
$$\frac{\partial \overline{u}}{\partial y} = \frac{\overline{u_2} - \overline{u_1}}{y_2 - y_1} \qquad (53)$$

The results of these calculations are listed in Table 2. The local transverse distances are close enough, so, the above approximation is quite reasonable. From analytical point of view, Eq. 34 is differentiated to give the shear stress distribution. The result of such differentiation gives the following relationship for shear stress distribution,

$$\tau_{Ana} = \frac{\upsilon}{\delta}\frac{u}{u^{*2}}(\frac{1}{7\eta} - \frac{4}{1-\eta}) \qquad (54)$$

The values of shear stress at different transverse locations are calculated based on the above equation and listed in Table 2. The two sets of shear stresses are plotted against the dimensionless transverse distance from the plate in Fig. 4.

Figure 4: *Analytical and experimental shear stress distributions in turbulent natural convection from a vertical heated plate*



4. Restrictions

The analytical solution technique for laminar and turbulent free convection from a vertical plate is based on some simplifying assumptions. The implications and limitations of the underlying assumptions should be addressed, as they are important issues. The incompressible flow assumption and negligible viscous work for air stream at very low velocity (M<0.3) is quite reasonable. The discrepancy between the analytical velocity profile and the existing experimental results in the special case of pure free convection could be because of power law velocity distribution. A different velocity profile may bring the analytical results closer to the experimental findings.

There is a limitation for applying this solution method to other fluids, because of assuming that $\delta_T \approx \delta$. This assumption is valid for fluids with Pr=1, however for a Prandtl number less than one (Pr for air $\approx$0.7) the thermal boundary layer is thicker than the velocity boundary layer.

5. Discussion

Analytical solutions were applied to laminar and turbulent free convection from isothermal vertical heated plate. In the laminar boundary layer, the mean Nusselt number was compared with the exact solution calculated numerically by Schmidt and Beckman [8]. The difference was only 5 percent.

In the turbulent boundary layer the velocities, shear stresses, temperatures and heat transfer coefficients were compared with the results obtained experimentally by Abu-Mulaweh et al [5]. The average difference between the analytical and experimental temperature profiles is 9.7 percent. The average difference between the analytical and experimental Nusselt numbers is 6.4 percent.

The experimental velocities are about one order of magnitude bigger than the analytical ones. The experimental shear stresses except at point 3 mm away from the heated plate (very close to the wall) do not agree with the analytical shear stresses. The reason for this discrepancy may lie in the assumption that the thicknesses of thermal and velocity boundary layers have been considered to be equal. The experimental results of Abu-Mulaweh et al [5] show that these two thicknesses are not the same. In fact, from Table 2, at y=0.2 m, we have:

$$U_{Exp}=0.009, \quad \theta_{Exp}=0$$

This confirms that the velocity boundary layer is thinner than the thermal boundary layer. The other reason for disagreement between the experimental and analytical velocities and shear stresses could be because of assuming power law distribution for velocity. Alternatively, one may conclude that the experimental velocities measured in [5] are incorrect.

6. Conclusions

The polynomial profiles for velocity and temperature distributions in laminar free convection on a vertical flat plate are quite good approximations. The results for heat transfer coefficients based on the above approximation are only 5 percent different from the exact solution.

The power law profile for temperature distribution in turbulent free convection on a vertical flat plate is a satisfactory approximation. The results for heat transfer coefficients based on the above approximation are 6.4 percent different from the experimental results.

The power law profile for velocity distribution in turbulent free convection on a vertical plate is not a good approximation. The results for shear stresses based on the above approximation do not agree with the experimental data.

Table 2: *Analytical and experimental distributions of velocity, temperature, and shear stress in turbulent natural convection from a vertical heated plate*

$y(m)$	η	U_{Ana}	θ_{Ana}	$\tau_{Ana}\times10^5$	U_{Exp}	θ_{Exp}	$\tau_{Exp}\times10^5$
0.003	0.019108	0.53025	0.431854	12.250	0.224	0.529	10.865
0.004	0.025478	0.538285	0.408018	5.498	0.25	0.48	-51.374
0.005	0.031847	0.541334	0.388843	1.303	0.279	0.454	-80.213
0.006	0.038217	0.541141	0.372716	-1.548	0.293	0.371	-68.131
0.008	0.050955	0.534559	0.346399	-5.128	0.305	0.319	-120.968
0.01	0.063694	0.522835	0.325228	-7.212	0.3	0.279	-118.367
0.012	0.076433	0.508017	0.307422	-8.503	0.276	0.235	-119.997
0.015	0.095541	0.482397	0.284988	-9.600	0.261	0.244	-141.405
0.02	0.127389	0.435493	0.254991	-10.251	0.251	0.226	-178.551
0.025	0.159236	0.387471	0.230859	-10.169	0.25	0.179	-169.036
0.03	0.191083	0.340777	0.210563	-9.723	0.228	0.177	-184.117
0.04	0.254777	0.255768	0.177443	-8.358	0.191	0.153	-210.149
0.05	0.318471	0.184707	0.1508	-6.806	0.178	0.139	-186.387
0.06	0.382166	0.12804	0.128391	-5.310	0.158	0.117	-195.560
0.08	0.509554	0.052975	0.091824	-2.836	0.118	0.086	-144.137
0.1	0.636943	0.016423	0.062407	-1.205	0.086	0.045	-91.6566
0.125	0.796178	0.001684	0.032037	-0.223	0.065	0.024	-86.795
0.15	0.955414	0.0000039	0.006495	-0.002	0.042	0.01	-56.786
0.175					0.018	0.003	-30.707
0.2					0.009	0	-30.304

References

[1] Abu-Mulaweh, H. I., Armaly, B. F., & Chen, T. S., "Measurements of laminar mixed convection adjacent to vertical plate – Uniform wall heat flux case", J. Heat Transfer, vol. 114, no. 4, pp. 105710-59, 1992.

[2] Qiu, S., Simon, T. W., & Volino, R. J., "Evaluation of local wall temperature, heat flux, and convective heat transfer coefficient from the near wall temperature profile", ASME HTD-vol. 318, pp. 45-52, 1995.

[3] Patel, K., Armaly, B. F., & Chen, T. S., "Transition from turbulent natural to turbulent forced convection adjacent to isothermal vertical plate", HTD-vol. 324, Natl. Heat Transfer Conf., Houston, TX, vol. 2, pp. 51-56, 1996.

[4] Abu-Mulaweh, H. I., Armaly, B. F., & Chen, T. S., "Measurements of turbulent natural convection flow over a vertical backward-facing step", HTD-vol. 338, Proc. ASME Heat Transfer Division, IMECE, Atlanta, GA, vol. 2, pp. 313-321, 1996.

[5] Abu-Mulawen, H. I., Chen, T. S., & Armaly B. F., "Effects of free stream velocity on turbulent natural convection flow along a vertical plate", Experimental Heat Transfer, an International Journal, vol. 13, No. 3, pp. 183-195, 2002.

[6] Bayley, F. J., Owen, J. M., & Turner, A. B., "Heat transfer", Thomas Nelson and sons Ltd., 0 17 7616609 1, 1972.

[7] Rohsenow, W. M., & Choi, H. " Heat, mass, and momentum transfer", Prentice-Hall, Inc., 1961.

[8] Schmidt, E., & Beckmann, Forsch. Gebiete Ingenieurw., 1, 391, 1930.

[9] Eckert, E. R. G., & Jackson, T. W., "Analysis of turbulent free convection boundary layer on a flat plate", NACA Rept. 1015, 1951.

[10] Blasius, H., "Das Ahnlichkeitsgesetz bei Reibungsvorgangen in Flussigkeiten", Forsch. Arb. IngWes., No. 131, 1913.

[11] Reynolds, O., "On the extent and action of the heating surface for steam boilers", Proc. Manchester Lit. Phil. Soc., 14, 7, 1874.